Regularized moment equations for binary gas mixtures: Derivation and linear analysis

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The applicability of the order of magnitude method [H. Struchtrup, "Stable transport equations for rarefied gases at high orders in the Knudsen number", Phys. Fluids **16**, 3921–3934 (2004)] is extended to binary gas mixtures in order to derive various sets of equations—having minimum number of moments at a given order of accuracy in the Knudsen number—for binary mixtures of monatomic-inert-ideal gases interacting with the Maxwell interaction potential. For simplicity, the equations are derived in the linear regime up to third order accuracy in the Knudsen number. At zeroth order, the method produces the Euler equations; at first order, it results into the Fick, Navier–Stokes, and Fourier equations; at second order, it yields a set of 17 moment equations; and at third order it leads to the regularized 17-moment equations. The transport coefficients in the Fick, Navier–Stokes, and Fourier equations obtained through order of magnitude method are compared with those obtained through the classical Chapman–Enskog expansion method. It is established that the different temperatures of different constituents do not play a role up to second order accurate theories in the Knudsen number, whereas they do contribute to third order accurate theory in the Knudsen number. Furthermore, it is found *empirically* that the zeroth, first and second order accurate equations are linearly stable for all binary gas mixtures; however, although the third order accurate regularized 17-moment equations are linearly stable for most of the mixtures, they are linearly unstable for mixtures having extreme difference in molecular masses.

Keywords: Regularized moment equations, Boltzmann equation, binary gas mixtures, order of magnitude method, Grad's moment equations.

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I. INTRODUCTION

It is well-established that the Navier–Stokes and Fourier equations break down in describing non-equilibrium processes in rarefied gases since they typically lie outside the hydrodynamic regime.[1–4] The flow regime is, usually, identified by a dimensionless parameter, the Knudsen number (Kn) which is defined as the ratio of mean free path of gas molecules to a characteristic length scale pertaining to the problem. Processes in all flow regimes, i.e., for all Knudsen numbers, can be well-described by the Boltzmann equation(s);[1–5] nevertheless, the direct numerical solutions[6,7] of the Boltzmann equation(s) or the solutions obtained with direct simulation Monte Carlo (DSMC) method[5] are computationally very expensive, particularly in the early transition regime ($0.05 \lesssim \mathrm{Kn} \lesssim 1$). Since many processes encountered in practical problems (such as processes in microscale flows) beset in this regime, there is a crave for accurate and efficient models which are capable of computing rarefied processes—particularly in the transition regime—with less computational cost.

These models, usually, emanate from the Boltzmann equation(s) through approximation techniques in kinetic theory. The two classical and most avowed approximation techniques in kinetic theory of single gases are the *Chapman–Enskog expansion method*[1,3,8–11] and *Grad's method of moments*.[12–14] Both methods can be found in standard textbooks, e.g., Refs. 2, 3, 8, 10, 15–17.

The Chapman–Enskog expansion method is applicable to processes, which are close to equilibrium ($\mathrm{Kn} \to 0$). The method relies on an asymptotic analysis in powers of the Knudsen number. In this method, the velocity distribution function of gas molecules is expanded in powers of the Knudsen number. This expansion for the velocity distribution function is then inserted into the Boltzmann equation and the coefficients of each power of the Knudsen number are compared on both sides of the equation. The procedure leads to the constitutive relations of different orders for the well-known conservation laws of fluid dynamics. At zeroth order, the method gives the Euler equations; at first order, it yields the classical Navier–Stokes and Fourier equations; at second order, it results into the Burnett equations; at third order, it leads to the so-called super-Burnett equations, and so on. The super-Burnett equations are already so involved that the full super-Burnett equations do not seem to exist in present day literature. Besides complex structure due to the presence of higher order derivatives, the Burnett equations are known to suffer from inherent (linear) instabilities;[18] consequently, their use is not recommended.

In Grad's method of moments, the Boltzmann equation is supplanted by a system of first order partial differential equations, referred to as *moment equations*. Moment equations form an infinite set of coupled first order partial differential equations, which is not closed. Grad's method of moments truncates this infinite set at a certain level. Moreover, to close the set at this level, it approximates the velocity distribution function by an expansion in orthogonal polynomials—usually, Hermite polynomials—in (peculiar) velocity, and the coefficients in the expansion are obtained by satisfying the definition of the moments considered at that level. The moment equations resulting from Grad's method of moments (in case of single gases) are always linearly stable.[18] Unfortunately, the method does not, a priori, grant the touchstone on which and how many moments need to be considered for describing a process with a given Knudsen number. However, it can be stated empirically that the number of moments considered ought to be increased with increasing Knudsen number.[14,19] Furthermore, due to their hyperbolic nature, the well-known Grad's 13-moment (G13) equations for a single gas obtained via Grad's method of moments manifest non-physical sub-shocks for

flows with Mach numbers above 1.65[14,20] and do not capture Knudsen boundary layers.[21,22] Nevertheless, by considering more moments, Knudsen boundary layers can be captured[21,23] and smooth shock structure can be obtained for higher Mach numbers.[20]

In order to surmount the deficiencies inherent to both Chapman–Enskog expansion method and Grad's method of moments, Struchtrup and Torrilhon[24]—for single gases—introduced a new method, often referred to as the regularized moment method, which regularizes the original G13 equations for a single gas by means of a Chapman–Enskog expansion of Grad's 26-moment (G26) equations around a pseudo-equilibrium and leads to the regularized 13-moment (R13) equations. The R13 equations retain the enviable features of both the Chapman–Enskog expansion method and the Grad's method of moments while avert their shortcomings. The R13 equations are always linearly stable and engender smooth shock structures for all Mach numbers.[3,24]

For single gases, Struchtrup[25] employed another method, termed as order of magnitude method, and rederived the R13 equations. The order of magnitude method accounts for the order of magnitude of all moments and of each term present in moment equations in powers of the Knudsen number and was originally developed for studying "consistent order extended thermodynamics (COET)" by Müller, Reitebuch, and Weiss[26]; nonetheless, the approach of applying the order of magnitude method in Ref. 25 is quite different from that in Ref. 26. The method of Struchtrup[25] provides highly accurate equations and, concurrently, resolves the issue of how many moments need to be considered for describing a process with certain accuracy. The method has been applied initially to the Bhatnagar–Gross–Krook (BGK) model[27] as well as for the Maxwell interaction potential in Refs. 3 and 25 and, subsequently, also to the hard-sphere interaction potential in Ref. 28. For the BGK model and Maxwell and hard-sphere interaction potentials, the equations have been derived up to third order accuracy in the Knudsen number by exploiting the order of magnitude method,[25,28] where it yields Euler equations at zeroth order, Navier–Stokes equations at first order, G13 equations (without a non-linear term) for the BGK model and Maxwell interaction potential and a variant of G13 equations for the hard-sphere interaction potential at second order, and a variant of the original R13 equations[24] at third order. However, for general interaction potentials, the method has been employed to derive equations only up to second order accuracy[29] so far. Since their derivation, the R13 equations have been successfully employed to describe several processes in rarefied gases, see e.g., Refs. 30–39.

Unlike single gases, kinetic theory for gaseous mixtures is still not very mature. The PhD theses of Enskog[40] and Kolodner[41] can be regarded as the pioneering works on the Chapman–Enskog expansion method and Grad's method of moments, respectively, for gas mixtures. Ref. 42 describes the detailed derivation of Grad's moment system (especially, considering 13 moments for each component) for gas mixtures; nevertheless, the explicit expressions for the right-hand sides of these equations are computed by employing various approximations. In Refs. 43 and 44, the authors consider the moment equations for gas mixtures in the context of extended thermodynamics but use simplified models for computing the collision terms in these equations. Ref. 45 studies Grad's method of moments in a multi-component approach for plasma models by considering 13 moments for each constituent. Ref. 46 discusses the higher order Grad-type moment equations too, however, it does not include—for example—the third rank tensors. Furthermore, the Grad-type moment equations in both Refs. 45 and 46, see also the textbook 47, are derived based on linearized Boltzmann collision operators. In addition to this, Refs. 43–47 derive the moment equations by assuming a single average temperature for the whole mixture, however,

a multi-temperature description of gas mixtures, which considers different temperatures for different constituents in the mixture, is imperative for many practical problems,[48] especially for problems arising in plasma physics. Although, Ref. 43 and the textbook 47 discuss the multi-temperature approach, they promptly switch to the single temperature approach owing to simplicity.

Similar to a single gas case where the derivation of the R13 equations for a single gas requires G26 equations, the derivation of regularized moment equations for a gaseous mixture also requires higher order moment equations, and owing to the unavailability of higher order moment equations for gaseous mixtures until recently,[49] the regularization for gas mixtures has never been attempted before. In Ref. 49, two authors of the present paper have derived the fully non-linear G26 equations for each constituent in a mixture of gases interacting with the Maxwell interaction potential based on multi-temperature approach. Furthermore, the first author of the present paper has also extended the derivation of the fully non-linear G26 equations for each constituent in a mixture of gases based on multi-temperature approach to the hard-sphere interaction potential.[50] It is worth pointing out that the computation of Boltzmann collision integrals or production terms appearing in these equations is quite involved, particularly with the multi-temperature approach, and a detailed computational strategy for evaluation of the Boltzmann collision integrals associated with these equations can be found in Refs. 50–52.

In this paper, we exploit the G26 equations for each constituent in a gas mixture as detailed in Ref. 49, and derive various sets of equations up to third order accuracy in the Knudsen number by extending the applicability of the order of magnitude method to binary gas mixtures. For simplicity, in this paper, we focus our attention only to processes in *binary* mixtures of monatomic-inert-ideal gases interacting with the *Maxwell* interaction potential and in the *linear* regime. The derivation of similar systems of moment equations valid in the non-linear regime and, also, their derivation for other interaction potentials is beyond the scope of the present paper and will be considered elsewhere in the future. At zeroth order accuracy, the method gives the (linearized) Euler equations for binary gas-mixtures; at first order accuracy, it yields the (linearized) Fick, Navier–Stokes, and Fourier equations; at second order accuracy, it leads to the (linearized) 17 moment equations; and at third order accuracy, it results into the regularized 17-moment (R17) equations in linearized form. The Fick, Navier–Stokes, and Fourier laws obtained here are compared with those obtained via the classical Chapman–Enskog expansion method. Furthermore, the linear stability of the derived sets of equations is analyzed. However, the shock wave problems, H-theorem and boundary conditions for these equations are also out of scope of the present paper and will be considered elsewhere in the future.

The remainder of the paper is organized as follows. The conservation laws for a gas mixture are stated and problem is formulated in Section II. The order of magnitude method is adumbrated in Section III. Grad-type 26-moment equations for each constituent in a binary mixture of gases interacting with the Maxwell interaction potential are presented in linear-dimensionless form in Section IV. The order of magnitude method is employed to determine the magnitude of all non-equilibrium moments in Section V. The minimum moments to describe a process with a certain accuracy in the Knudsen number are identified in Section VI. In Section VII, moment equations with different orders of accuracy are derived, i.e., the Euler equations, Fick, Navier–Stokes, and Fourier equations, second order accurate equations, and, finally, the R17 equations for binary gas mixtures are derived. The linear stability of these equation is analyzed in Section VIII. The final conclusion and discussion are given in

Section X.

II. PROBLEM DESCRIPTION

The conservation laws for a mixture of monatomic-inert-ideal gases in absence of any external forces read[1,8,10,49]

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho_\alpha v_i + \rho_\alpha u_i^{(\alpha)}\right) = 0 \quad \forall \quad \alpha, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0, \tag{2}$$

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho v_i v_j + \sigma_{ij} + p\,\delta_{ij}\right) = 0, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p + \frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_i}\left[\left(\frac{3}{2}p + \frac{1}{2}\rho v^2\right) v_i + q_i + \left(\sigma_{ij} + p\,\delta_{ij}\right) v_j\right] = 0, \tag{4}$$

where

$$\rho = \sum_\alpha \rho_\alpha, \quad p = k_B\, n\, T = k_B \sum_\alpha n_\alpha T_\alpha, \quad n = \sum_\alpha n_\alpha, \quad \sigma_{ij} = \sum_\alpha \sigma_{ij}^{(\alpha)}, \quad \text{and} \quad q_i = \sum_\alpha q_i^{(\alpha)}$$

are the total mass density, total pressure, total number density, total stress and total heat flux of the mixture, respectively, with k_B being the Boltzmann constant, T being the average temperature of the mixture and α denoting one constituent in the mixture; moreover, $\rho_\alpha = m_\alpha n_\alpha$ is mass density of the constituent α with m_α being the molecular mass of species α and n_α being the number density of species α, and $u_i^{(\alpha)}$ is the diffusion velocity of the α-constituent in the mixture (see Ref. 49 for the definition). Eqs. (1) are the mass balance equations for individual species in the mixture, and Eqs. (2)–(4) are the mass balance, momentum balance and energy balance equations for the mixture, respectively. In fact, Eq. (2) is obtained by summing Eqs. (1) over all α's; note that the diffusion velocities in a gas mixture are not independent and they are related via

$$\sum_\alpha \rho_\alpha u_i^{(\alpha)} = 0. \tag{5}$$

Therefore, Eq. (1) for any one α can be dropped from the system of conservation laws (1)–(4) or, equivalently, Eq. (2) can be dropped from the system when including Eq. (1) for all α's.

Clearly, the system of conservation laws (1)–(4) is not closed, since it contains the unknowns: diffusion velocities $u_i^{(\alpha)}$, stress σ_{ij} and heat flux q_i. Therefore, in order to close the system of conservation laws (1)–(4), one must supply the constitutive equations for diffusion velocities $u_i^{(\alpha)}$, stress σ_{ij} and heat flux q_i. Here, we shall first determine the magnitudes of diffusion velocities $u_i^{(\alpha)}$, stress σ_{ij} and heat flux q_i in powers of the Knudsen number, and then systematically obtain the closed systems of equations in such a way that the unknowns $u_i^{(\alpha)}$, σ_{ij} and q_i in the conservation laws (1)–(4) are known up to a certain order in powers of the Knudsen number. We emphasize that in this paper, we shall focus only on binary mixtures of gases interacting with the Maxwell interaction potential and only in the linear regime.

III. OUTLINE OF ORDER OF MAGNITUDE METHOD

The order of magnitude method for finding the proper equations with order of accuracy λ_0 in the Knudsen number comprises of the following three steps.[3,25]

1. Determination of *order of magnitude* λ of the moments:

   The goal at this step is to determine the order of magnitude of moments in powers of a smallness parameter (ε) which is usually the Knudsen number. To this end, a (non-conserved) moment ϕ is expanded in powers of ε as

   $$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots$$

   It should be noticed that the above expansion performed on a moment ϕ is somewhat similar to the classical Chapman–Enskog expansion, which is performed on the velocity distribution function. However, unlike the approach of the classical Chapman–Enskog expansion which aims at computing ϕ_i's ($i = 0, 1, 2, \dots$), the focus in this method is just to determine the leading order of ϕ. The leading order of ϕ is determined by inserting the above expansion into the complete set of moment equations. A moment ϕ is said to be of leading order λ if $\phi_i = 0$ for all $i < \lambda$ and $\phi_\lambda \neq 0$. The leading order of a moment is the order of magnitude of that moment.

2. Construction of a system of moment equations having minimum number of moments at a given order of accuracy λ:

   At this step, some of the originally chosen moments are combined linearly in order to introduce new variables in the system. The new variables are constructed in such a way that on replacing the original moments in the moment equations with the new variables, the number of moments at a given order λ is minimum. This step not only provides an unambiguous set of moments at order λ but also guarantees that the final equations will be independent of the initial choice of moments.

3. Deletion of all terms in all equations that would lead to contributions of orders $\lambda > \lambda_0$ in the conservation laws:

   At this step, we adopt the following definition of the order of accuracy λ_0.

   Definition 1. *A set of equations for binary gas mixtures is said to be accurate of order λ_0, when the diffusion velocities (of both the components), total stress and total heat flux in the mixture are known up to order $\mathcal{O}(\varepsilon^{\lambda_0})$.*

   The adoption of this definition relies on the fact that all moment equations are strongly coupled. This connotes that each term in any of the moment equations has some influence on all other equations, particularly on the conservation laws. The influence of each term can be weighted by some power in the Knudsen number, and is related—but not equal to—the order of magnitude of the moments present in that term. A theory of order λ_0 considers only those terms—in all the equations—whose leading order of influence in the conservation laws is $\lambda \leq \lambda_0$, and the terms not fulfilling this condition are simply ignored. In order to apply this condition, it suffices to start with the conservation laws, and add the relevant terms step-by-step, order-by-order. We start with order $\mathcal{O}(\varepsilon^0)$ equations (Euler), then add the relevant terms to obtain order $\mathcal{O}(\varepsilon^1)$ equations (Fick, Navier–Stokes, and Fourier equations) and so on.

IV. LINEAR-DIMENSIONLESS EQUATIONS

Since we shall derive the equations valid in the *linear* regime, it is more convenient to use the linear-dimensionless variables. For linearization and non-dimensionlization of the variables, the reader is referred to Refs. 49 and 50.

A. Conservation laws

The conservation laws for a binary mixture of gases α and β in linear-dimensionless form read

$$\varkappa_\alpha\left(\frac{\partial \hat{n}_\alpha}{\partial \hat{t}}+\frac{\partial \hat{v}_i}{\partial \hat{x}_i}\right)+\frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i}=0, \tag{6}$$

$$\varkappa_\beta\left(\frac{\partial \hat{n}_\beta}{\partial \hat{t}}+\frac{\partial \hat{v}_i}{\partial \hat{x}_i}\right)+\frac{\partial \hat{u}_i^{(\beta)}}{\partial \hat{x}_i}=0, \tag{7}$$

$$\frac{\partial}{\partial \hat{t}}\left(\mu_\alpha \mathrm{x}_\alpha^\circ \hat{n}_\alpha+\mu_\beta \mathrm{x}_\beta^\circ \hat{n}_\beta\right)+\left(\mu_\alpha \mathrm{x}_\alpha^\circ+\mu_\beta \mathrm{x}_\beta^\circ\right)\frac{\partial \hat{v}_i}{\partial \hat{x}_i}=0, \tag{8}$$

$$\varkappa\frac{\partial \hat{v}_i}{\partial \hat{t}}+\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j}+\frac{\partial}{\partial \hat{x}_i}\left(\mathrm{x}_\alpha^\circ \hat{n}_\alpha+\mathrm{x}_\beta^\circ \hat{n}_\beta\right)+\frac{\partial \hat{T}}{\partial \hat{x}_i}=0, \tag{9}$$

$$\frac{3}{2}\frac{\partial \hat{T}}{\partial \hat{t}}+\frac{\partial \hat{v}_i}{\partial \hat{x}_i}+\frac{\partial \hat{h}_i}{\partial \hat{x}_i}+\frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha}\frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i}+\frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta}\frac{\partial \hat{u}_i^{(\beta)}}{\partial \hat{x}_i}=0. \tag{10}$$

While writing Eqs. (6)–(10), the abbreviations

$$\varkappa_\alpha=\frac{v_\circ}{\sqrt{\theta_\alpha^\circ}},\quad \varkappa_\beta=\frac{v_\circ}{\sqrt{\theta_\beta^\circ}},\quad \varkappa=\mathrm{x}_\alpha^\circ\varkappa_\alpha^2+\mathrm{x}_\beta^\circ\varkappa_\beta^2 \tag{11}$$

have been used; here $v_\circ$ is a velocity scale, and $\theta_\alpha^\circ=k_BT_\circ/m_\alpha$ and $\theta_\beta^\circ=k_BT_\circ/m_\beta$ are the ground state temperatures of α- and β-species in energy units with m_α and m_β being the molecular masses of species α and β. Moreover, in Eqs. (6)–(10),

$$\mu_\alpha=\frac{m_\alpha}{m_\alpha+m_\beta}\quad\text{and}\quad \mu_\beta=\frac{m_\beta}{m_\alpha+m_\beta} \tag{12}$$

are the mass ratios of α- and β-constituents in the mixture, respectively, and these notations for the mass ratios are adopted following the textbook 9;

$$\mathrm{x}_\alpha^\circ=\frac{n_\alpha^\circ}{n_\circ}\quad\text{and}\quad \mathrm{x}_\beta^\circ=\frac{n_\beta^\circ}{n_\circ} \tag{13}$$

with $n_\circ = n_\alpha^\circ + n_\beta^\circ$ are the mole fractions of α- and β-constituents in the ground state, respectively;

$$\hat{T} = \mathrm{x}_\alpha^\circ\, \hat{T}_\alpha + \mathrm{x}_\beta^\circ\, \hat{T}_\beta, \quad \hat{\sigma}_{ij} = \mathrm{x}_\alpha^\circ\, \hat{\sigma}_{ij}^{(\alpha)} + \mathrm{x}_\beta^\circ\, \hat{\sigma}_{ij}^{(\beta)} \quad \text{and} \quad \hat{h}_i = \frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha}\hat{h}_i^{(\alpha)} + \frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta}\hat{h}_i^{(\beta)} \tag{14}$$

are the dimensionless perturbations in average temperature, total stress and total reduced heat flux of the mixture from their respective ground state values with

$$\hat{h}_i^{(\alpha)} = \hat{q}_i^{(\alpha)} - \frac{5}{2}\hat{u}_i^{(\alpha)} \quad \text{and} \quad \hat{h}_i^{(\beta)} = \hat{q}_i^{(\beta)} - \frac{5}{2}\hat{u}_i^{(\beta)} \tag{15}$$

being the dimensionless perturbations to the reduced heat fluxes of species α and β,[46,47,50] respectively; $\hat{t}$ and $\hat{x}$ denote the dimensionless time and dimensionless space, respectively; and all other quantities with hats denote the dimensionless perturbations from their respective ground state values. Here, the total stress (σ_{ij}) and the total reduced heat flux (h_i) are scaled as

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{k\, n_\circ T_\circ} \quad \text{and} \quad \hat{h}_i = \frac{h_i}{k\, n_\circ T_\circ v_\circ},$$

see Ref. 50. It should be noted that Eq. (8) can be obtained from Eqs. (6) and (7), thus Eqs. (6)–(8) are not independent.

B. Grad-type moment equations

In the light of definition 1, our goal is to obtain various sets of equations in such a way that $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$ in the conservation laws (6)–(10) are known up to a certain order in powers of the Knudsen number. To this end, we require the extended Grad-type moment equations in linear-dimensionless form, which, for the Maxwell interaction potential, have been derived in detail in Ref. 49 (Eqs. (4.3)–(4.10) of Ref. 49 for both the constituents). Here, we shall use them directly but rename the Knudsen number from Kn to ε in them, and instead of the field variables $\hat{T}_\alpha$, $\hat{T}_\beta$, $\hat{q}_i^{(\alpha)}$, $\hat{q}_i^{(\beta)}$, $\hat{u}_{ij}^{1(\alpha)}$, $\hat{u}_{ij}^{1(\beta)}$, we shall write them in the new field variables, $\hat{T}$, $\Delta\hat{T}$, $\hat{h}_i^{(\alpha)}$, $\hat{h}_i^{(\beta)}$, $\hat{R}_{ij}^{1(\alpha)}$, $\hat{R}_{ij}^{1(\beta)}$, where $\hat{h}_i^{(\alpha)}$, $\hat{h}_i^{(\beta)}$ are given by Eqs. (15),

$$\hat{R}_{ij}^{(\alpha)} = \hat{u}_{ij}^{1(\alpha)} - 7\,\hat{\sigma}_{ij}^{(\alpha)}, \quad \hat{R}_{ij}^{(\beta)} = \hat{u}_{ij}^{1(\beta)} - 7\,\hat{\sigma}_{ij}^{(\beta)}, \tag{16}$$

and

$$\Delta\hat{T} = \hat{T}_\alpha - \hat{T}_\beta \tag{17}$$

is the dimensionless perturbation in the temperature difference from its ground state value. Nevertheless, the equations for the new field variables can be obtained from Eqs. (4.3)–(4.10) of Ref. 49 for both the constituents in a straightforward way by combining them linearly, and therefore, the details are omitted here for the sake of conciseness. The advantages of using the new field variables—in case of the Maxwell interaction potential—are: (i) it will be seen below that although $\hat{T}_\alpha$, $\hat{T}_\beta$ and $\hat{T}$ are the zeroth order quantities, $\Delta\hat{T}$ will be a second order quantity; consequently, it will not play a role for the theories up to second

order in the Knudsen number, (*ii*) the use of $\hat{h}_i^{(\alpha)}$ and $\hat{h}_i^{(\beta)}$ decouples the right-hand sides of their governing equations from the diffusion velocities $\hat{u}_i^{(\alpha)}$ and $\hat{u}_i^{(\beta)}$ (see Eqs. (23) and (24) below), and (*iii*) the use of $\hat{R}_{ij}^{(\alpha)}$ and $\hat{R}_{ij}^{(\beta)}$ decouples the right-hand sides of their governing equations from the individual stresses $\hat{\sigma}_{ij}^{(\alpha)}$ and $\hat{\sigma}_{ij}^{(\beta)}$ (see Eqs. (27) and (28) below). It is emphasized that even if one does not change $\hat{q}_i^{(\alpha)}$, $\hat{q}_i^{(\beta)}$, $\hat{u}_{ij}^{1(\alpha)}$ and $\hat{u}_{ij}^{1(\beta)}$ to the new variables at this point, they will automatically be combined linearly at the second step of the order of magnitude method in order to produce exactly the same results as below.

The system of linear-dimensionless extended Grad-type moment equations for a mixture of gases α and β, which is equivalent to the system of Eqs. (4.3)–(4.10) of Ref. 49 for both the constituents, in the new field variables includes the individual mass balance equations (6) and (7), the energy balance equation for the mixture (10), and the moment equations

$$\varkappa_\alpha \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{t}} + \frac{\mathrm{x}_\beta^\circ}{\varkappa}\left[\varkappa_\beta^2\left(\frac{\partial \hat{\sigma}_{ij}^{(\alpha)}}{\partial \hat{x}_j} + \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i}\right) - \varkappa_\alpha^2\left(\frac{\partial \hat{\sigma}_{ij}^{(\beta)}}{\partial \hat{x}_j} + \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i}\right)\right] - \frac{\mathrm{x}_\beta^\circ}{\varkappa}(\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_i} + \mathrm{x}_\beta^\circ \frac{\partial \Delta\hat{T}}{\partial \hat{x}_i} = -\delta_1 \frac{1}{\varepsilon\,\Omega}\mathrm{x}_\beta^\circ\left(\hat{u}_i^{(\alpha)} - \frac{\varkappa_\alpha}{\varkappa_\beta}\hat{u}_i^{(\beta)}\right), \tag{18}$$

$$\varkappa_\beta \frac{\partial \hat{u}_i^{(\beta)}}{\partial \hat{t}} + \frac{\mathrm{x}_\alpha^\circ}{\varkappa}\left[\varkappa_\alpha^2\left(\frac{\partial \hat{\sigma}_{ij}^{(\beta)}}{\partial \hat{x}_j} + \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i}\right) - \varkappa_\beta^2\left(\frac{\partial \hat{\sigma}_{ij}^{(\alpha)}}{\partial \hat{x}_j} + \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i}\right)\right] - \frac{\mathrm{x}_\alpha^\circ}{\varkappa}(\varkappa_\beta^2 - \varkappa_\alpha^2)\frac{\partial \hat{T}}{\partial \hat{x}_i} - \mathrm{x}_\alpha^\circ \frac{\partial \Delta\hat{T}}{\partial \hat{x}_i} = -\gamma_1 \frac{1}{\varepsilon\,\Omega}\mathrm{x}_\alpha^\circ\left(\hat{u}_i^{(\beta)} - \frac{\varkappa_\beta}{\varkappa_\alpha}\hat{u}_i^{(\alpha)}\right), \tag{19}$$

$$\frac{3}{2}\frac{\partial \Delta\hat{T}}{\partial \hat{t}} + \frac{1}{\varkappa_\alpha}\left(\frac{\partial \hat{h}_i^{(\alpha)}}{\partial \hat{x}_i} + \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i}\right) - \frac{1}{\varkappa_\beta}\left(\frac{\partial \hat{h}_i^{(\beta)}}{\partial \hat{x}_i} + \frac{\partial \hat{u}_i^{(\beta)}}{\partial \hat{x}_i}\right) = -\frac{1}{\varepsilon\,\Omega}\frac{\delta_2}{\varkappa_\alpha}\Delta\hat{T}, \tag{20}$$

$$\varkappa_\alpha\left(\frac{\partial \hat{\sigma}_{ij}^{(\alpha)}}{\partial \hat{t}} + 2\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right) + \frac{\partial \hat{m}_{ijk}^{(\alpha)}}{\partial \hat{x}_k} + \frac{4}{5}\frac{\partial \hat{h}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} + 2\frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left\{\mathrm{x}_\alpha^\circ\Omega_\alpha\hat{\sigma}_{ij}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_3\hat{\sigma}_{ij}^{(\alpha)} - \delta_4\hat{\sigma}_{ij}^{(\beta)}\right)\right\}, \tag{21}$$

$$\varkappa_\beta\left(\frac{\partial \hat{\sigma}_{ij}^{(\beta)}}{\partial \hat{t}} + 2\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right) + \frac{\partial \hat{m}_{ijk}^{(\beta)}}{\partial \hat{x}_k} + \frac{4}{5}\frac{\partial \hat{h}_{\langle i}^{(\beta)}}{\partial \hat{x}_{j\rangle}} + 2\frac{\partial \hat{u}_{\langle i}^{(\beta)}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left\{\mathrm{x}_\beta^\circ\Omega_\beta\hat{\sigma}_{ij}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_3\hat{\sigma}_{ij}^{(\beta)} - \gamma_4\hat{\sigma}_{ij}^{(\alpha)}\right)\right\}, \tag{22}$$

$$\varkappa_\alpha \frac{\partial \hat{h}_i^{(\alpha)}}{\partial \hat{t}} + \frac{1}{2}\frac{\partial \hat{R}_{ij}^{(\alpha)}}{\partial \hat{x}_j} + \frac{\partial \hat{\sigma}_{ij}^{(\alpha)}}{\partial \hat{x}_j} + \frac{1}{6}\frac{\partial \hat{\Delta}_\alpha}{\partial \hat{x}_i} + \frac{5}{2}\frac{\partial \hat{T}}{\partial \hat{x}_i} + \frac{5}{2}\mathrm{x}_\beta^\circ \frac{\partial \Delta \hat{T}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{2}{3}\mathrm{x}_\alpha^\circ \Omega_\alpha \hat{h}_i^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_5 \hat{h}_i^{(\alpha)} - \delta_6 \hat{h}_i^{(\beta)}\right)\right\}, \tag{23}$$

$$\varkappa_\beta \frac{\partial \hat{h}_i^{(\beta)}}{\partial \hat{t}} + \frac{1}{2}\frac{\partial \hat{R}_{ij}^{(\beta)}}{\partial \hat{x}_j} + \frac{\partial \hat{\sigma}_{ij}^{(\beta)}}{\partial \hat{x}_j} + \frac{1}{6}\frac{\partial \hat{\Delta}_\beta}{\partial \hat{x}_i} + \frac{5}{2}\frac{\partial \hat{T}}{\partial \hat{x}_i} - \frac{5}{2}\mathrm{x}_\alpha^\circ \frac{\partial \Delta \hat{T}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{2}{3}\mathrm{x}_\beta^\circ \Omega_\beta \hat{h}_i^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_5 \hat{h}_i^{(\beta)} - \gamma_6 \hat{h}_i^{(\alpha)}\right)\right\}, \tag{24}$$

$$\varkappa_\alpha \frac{\partial \hat{m}_{ijk}^{(\alpha)}}{\partial \hat{t}} + \frac{3}{7}\frac{\partial \hat{R}_{\langle ij}^{(\alpha)}}{\partial \hat{x}_{k\rangle}} + 3\frac{\partial \hat{\sigma}_{\langle ij}^{(\alpha)}}{\partial \hat{x}_{k\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha \hat{m}_{ijk}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_7 \hat{m}_{ijk}^{(\alpha)} - \delta_8 \hat{m}_{ijk}^{(\beta)}\right)\right\}, \tag{25}$$

$$\varkappa_\beta \frac{\partial \hat{m}_{ijk}^{(\beta)}}{\partial \hat{t}} + \frac{3}{7}\frac{\partial \hat{R}_{\langle ij}^{(\beta)}}{\partial \hat{x}_{k\rangle}} + 3\frac{\partial \hat{\sigma}_{\langle ij}^{(\beta)}}{\partial \hat{x}_{k\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta \hat{m}_{ijk}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_7 \hat{m}_{ijk}^{(\beta)} - \gamma_8 \hat{m}_{ijk}^{(\alpha)}\right)\right\}, \tag{26}$$

$$\varkappa_\alpha \frac{\partial \hat{R}_{ij}^{(\alpha)}}{\partial \hat{t}} + 2\frac{\partial \hat{m}_{ijk}^{(\alpha)}}{\partial \hat{x}_k} + \frac{28}{5}\frac{\partial \hat{h}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{7}{6}\mathrm{x}_\alpha^\circ \Omega_\alpha \hat{R}_{ij}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_9 \hat{R}_{ij}^{(\alpha)} - \delta_{10} \hat{R}_{ij}^{(\beta)}\right)\right\}, \tag{27}$$

$$\varkappa_\beta \frac{\partial \hat{R}_{ij}^{(\beta)}}{\partial \hat{t}} + 2\frac{\partial \hat{m}_{ijk}^{(\beta)}}{\partial \hat{x}_k} + \frac{28}{5}\frac{\partial \hat{h}_{\langle i}^{(\beta)}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{7}{6}\mathrm{x}_\beta^\circ \Omega_\beta \hat{R}_{ij}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_9 \hat{R}_{ij}^{(\beta)} - \gamma_{10} \hat{R}_{ij}^{(\alpha)}\right)\right\}, \tag{28}$$

$$\varkappa_\alpha \frac{\partial \hat{\Delta}_\alpha}{\partial \hat{t}} + 8\frac{\partial \hat{h}_i^{(\alpha)}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{2}{3}\mathrm{x}_\alpha^\circ \Omega_\alpha \hat{\Delta}_\alpha + \mathrm{x}_\beta^\circ\left(\delta_{11} \hat{\Delta}_\alpha - \delta_{12} \hat{\Delta}_\beta\right)\right\}, \tag{29}$$

$$\varkappa_\beta \frac{\partial \hat{\Delta}_\beta}{\partial \hat{t}} + 8\frac{\partial \hat{h}_i^{(\beta)}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\,\Omega}\left\{\frac{2}{3}\mathrm{x}_\beta^\circ \Omega_\beta \hat{\Delta}_\beta + \mathrm{x}_\alpha^\circ\left(\gamma_{11} \hat{\Delta}_\beta - \gamma_{12} \hat{\Delta}_\alpha\right)\right\}. \tag{30}$$

In Eqs. (18)–(30), the coefficients $\delta_1, \delta_2, \ldots, \delta_{12}$ and $\gamma_1, \gamma_2, \ldots, \gamma_{12}$ depend only on the mass ratios μ_α and μ_β, and they are given in appendix A for better readability. Again, the field variables with hats denote the dimensionless perturbations from their respective ground state values; $\Omega = \mathrm{x}_\alpha^\circ\,\Omega_\alpha + \mathrm{x}_\beta^\circ\,\Omega_\beta$, where

$$\Omega_\alpha = \frac{\Omega_{\alpha\alpha}^{(2,2)}}{\Omega_{\alpha\beta}^{(2,2)}} \quad \text{and} \quad \Omega_\beta = \frac{\Omega_{\beta\beta}^{(2,2)}}{\Omega_{\alpha\beta}^{(2,2)}} \tag{31}$$

and $\Omega_{ij}^{(l,r)}$ are the standard Omega integrals;[10,49,50,52] finally,

$$\varepsilon = \frac{\ell}{L} \quad \text{with} \quad \ell = \frac{5}{16\sqrt{\pi}\, n_\circ \left(\mathrm{x}_\alpha^\circ\, \Omega_{\alpha\alpha}^{(2,2)} + \mathrm{x}_\beta^\circ\, \Omega_{\beta\beta}^{(2,2)} \right)} \tag{32}$$

is the Knudsen number; here, ℓ is the mean free path and L is the relevant macroscopic length scale.

C. Assumption about parameters

In the linear-dimensionless conservation laws (6)–(10) and in Grad-type moment equations (18)–(30), we find the parameters: μ_α, μ_β, x_α°, x_β°, Ω_α, Ω_β, $\varkappa_\alpha$, $\varkappa_\beta$ and ε. The parameters μ_α and μ_β are the mass ratios of the constituents and are given by Eqs. (12); x_α° and x_β° are the mole fractions of the constituents in ground state and are given by Eqs. (13); Ω_α and Ω_β are the ratios of Omega integrals, which are related to collision cross sections, and are given by Eqs. (31); $\varkappa_\alpha$ and $\varkappa_\beta$ are somewhat like inverse Mach numbers for each component in the mixture and are given by (11); finally, ε is the Knudsen number and is given by Eq. $(32)_1$. We assume that the parameters μ_α, μ_β, x_α°, x_β°, Ω_α, Ω_β, $\varkappa_\alpha$ and $\varkappa_\beta$ are of order $\mathcal{O}(1)$ in comparison to order of the Knudsen number, i.e., $\mathcal{O}(\varepsilon)$, otherwise one would have to consider the influence of these parameters in powers of the Knudsen number while performing the order of magnitude method and this would render the procedure extremely cumbersome. The assumption immediately excludes mixtures having large differences in molecular masses. We shall also see in Section VIII that without considering the influence of these parameters in powers of the Knudsen number (i.e., by assuming that μ_α, μ_β, x_α°, x_β°, Ω_α, Ω_β, $\varkappa_\alpha$ and $\varkappa_\beta$ are of order $\mathcal{O}(1)$ in comparison to order of the Knudsen number), the resulting R17 equations would be linearly unstable for mixtures having extreme differences in molecular masses.

V. THE ORDER OF MAGNITUDE OF MOMENTS

We shall now determine the orders of magnitude to the moments and then construct new sets of moments in such a way that we have minimum number of variables at each order.

In order to examine the order of magnitude of moments, we expand the non-conserved quantities (Ψ) in powers of the Knudsen number (ε) as

$$\Psi = \Psi_{|0} + \varepsilon\, \Psi_{|1} + \varepsilon^2 \Psi_{|2} + \ldots,$$

where $\Psi \in \{\Delta\hat{T}, \hat{u}_i^{(\alpha)}, \hat{u}_i^{(\beta)}, \hat{\sigma}_{ij}^{(\alpha)}, \hat{\sigma}_{ij}^{(\beta)}, \hat{h}_i^{(\alpha)}, \hat{h}_i^{(\beta)}, \hat{m}_{ijk}^{(\alpha)}, \hat{m}_{ijk}^{(\beta)}, \hat{R}_{ij}^{(\alpha)}, \hat{R}_{ij}^{(\beta)}, \hat{\Delta}_\alpha, \hat{\Delta}_\beta\}$, and the quantities $\Psi_{|0}$, $\Psi_{|1}$, $\Psi_{|2}$, … are of order $\mathcal{O}(\varepsilon^0)$. We substitute the above expansions in Eqs. (20)–(30) and compare the coefficients of each power of ε.

Comparing coefficients of ε^{-1} on both sides of Eqs. (20)–(30) one readily finds that $\Psi_{|0} = 0$ for all Ψ because there are no terms of order $\mathcal{O}(\varepsilon^{-1})$ on the left-hand sides of the balance equations for these quantities. This concludes that the leading orders of all the non-conserved quantities are at least one.

Comparing coefficients of ε^0 on both sides of Eqs. (20)–(30), it turns out that $\hat{u}_{i|1}^{(\alpha)}$, $\hat{u}_{i|1}^{(\beta)}$,

$\hat{n}_{i|1}^{(\alpha)}$, $\hat{\sigma}_{ij|1}^{(\beta)}$, $\hat{h}_{i|1}^{(\alpha)}$, $\hat{h}_{i|1}^{(\beta)}$ do not vanish whereas

$$\Delta\hat{T}_{|1} = \hat{m}_{ijk|1}^{(\alpha)} = \hat{m}_{ijk|1}^{(\beta)} = \hat{R}_{ij|1}^{(\alpha)} = \hat{R}_{ij|1}^{(\beta)} = \hat{\Delta}_{\alpha|1} = \hat{\Delta}_{\beta|1} = 0, \tag{33}$$

see appendix B for details. In other words, the leading orders of the diffusion velocities, stresses and heat fluxes of the both the constituents are one while the leading orders of temperature difference and other higher moments for both the constituents are at least two.

Comparing the coefficients of ε^1 on both sides of Eqs. (20) and (25)–(30), it turns out that none of $\Delta\hat{T}_{|2}$, $\hat{m}_{ijk|2}^{(\alpha)}$, $\hat{m}_{ijk|2}^{(\beta)}$, $\hat{R}_{ij|2}^{(\alpha)}$, $\hat{R}_{ij|2}^{(\beta)}$, $\hat{\Delta}_{\alpha|2}$, and $\hat{\Delta}_{\beta|2}$ vanish, see appendix B again for details. Therefore, the leading orders of all these quantities are two.

We shall not go further as the above is sufficient for obtaining the third order accurate (regularized) moment equations.

VI. MINIMUM NUMBER OF MOMENTS AT A GIVEN ORDER

A. Minimum number of moments of $\mathcal{O}(\varepsilon)$

We have established in Section III that $\hat{u}_i^{(\alpha)}$, $\hat{u}_i^{(\beta)}$, $\hat{\sigma}_{ij}^{(\alpha)}$, $\hat{\sigma}_{ij}^{(\beta)}$, $\hat{h}_i^{(\alpha)}$, $\hat{h}_i^{(\beta)}$ are the moments of order $\mathcal{O}(\varepsilon)$. In order to have minimum number of moments of order $\mathcal{O}(\varepsilon)$, let us first write down their leading order contributions (by solving Eqs. (B2)–(B4) of appendix B along with relation (5) in linear-dimensionless form for the binary mixture), which read

$$\hat{u}_{i|1}^{(\alpha)} = -\mathrm{x}_\beta^\circ \frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left[\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_i} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_i} - (\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_i}\right], \quad \hat{u}_{i|1}^{(\beta)} = -\frac{\varkappa_\alpha}{\varkappa_\beta}\frac{\mathrm{x}_\alpha^\circ}{\mathrm{x}_\beta^\circ}\hat{u}_{i|1}^{(\alpha)}, \tag{34}$$

$$\left.\begin{aligned}\hat{\sigma}_{ij|1}^{(\alpha)} &= -2\eta_\alpha\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}} \quad \text{where} \quad \eta_\alpha = \frac{\Omega\{\varkappa_\alpha(\mathrm{x}_\beta^\circ\Omega_\beta + \mathrm{x}_\alpha^\circ\gamma_3) + \varkappa_\beta\mathrm{x}_\beta^\circ\delta_4\}}{(\mathrm{x}_\alpha^\circ\Omega_\alpha + \mathrm{x}_\beta^\circ\delta_3)(\mathrm{x}_\beta^\circ\Omega_\beta + \mathrm{x}_\alpha^\circ\gamma_3) - \mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\gamma_4\delta_4},\\ \hat{\sigma}_{ij|1}^{(\beta)} &= -2\eta_\beta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}} \quad \text{where} \quad \eta_\beta = \frac{\Omega\{\varkappa_\beta(\mathrm{x}_\alpha^\circ\Omega_\alpha + \mathrm{x}_\beta^\circ\delta_3) + \varkappa_\alpha\mathrm{x}_\alpha^\circ\gamma_4\}}{(\mathrm{x}_\alpha^\circ\Omega_\alpha + \mathrm{x}_\beta^\circ\delta_3)(\mathrm{x}_\beta^\circ\Omega_\beta + \mathrm{x}_\alpha^\circ\gamma_3) - \mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\gamma_4\delta_4},\end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned}\hat{h}_{i|1}^{(\alpha)} &= -\kappa_\alpha\frac{\partial\hat{T}}{\partial\hat{x}_i} \quad \text{where} \quad \kappa_\alpha = \frac{\frac{5}{2}\Omega\left\{\left(\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta + \mathrm{x}_\alpha^\circ\gamma_5\right) + \mathrm{x}_\beta^\circ\delta_6\right\}}{\left(\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha + \mathrm{x}_\beta^\circ\delta_5\right)\left(\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta + \mathrm{x}_\alpha^\circ\gamma_5\right) - \mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\gamma_6\delta_6},\\ \hat{h}_{i|1}^{(\beta)} &= -\kappa_\beta\frac{\partial\hat{T}}{\partial\hat{x}_i} \quad \text{where} \quad \kappa_\beta = \frac{\frac{5}{2}\Omega\left\{\left(\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha + \mathrm{x}_\beta^\circ\delta_5\right) + \mathrm{x}_\alpha^\circ\gamma_6\right\}}{\left(\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha + \mathrm{x}_\beta^\circ\delta_5\right)\left(\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta + \mathrm{x}_\alpha^\circ\gamma_5\right) - \mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\gamma_6\delta_6}.\end{aligned}\right\} \tag{36}$$

Eqs. (34) are the Fick's law of diffusion (in linearized form) for the mixture; Eqs. (35) represent the laws somewhat similar to Navier–Stokes law for each component in the mixture; and Eqs. (36) represent the laws somewhat similar to Fourier's law for each components in the mixture.

As the diffusion velocities $\hat{u}_i^{(\alpha)}$ and $\hat{u}_i^{(\beta)}$ depend on each other, one can use any one of them in the moment equations. Moreover, the other first order quantities—the stresses $\hat{\sigma}_{ij}^{(\alpha)}$ and $\hat{\sigma}_{ij}^{(\beta)}$, and the reduced heat fluxes $\hat{h}_i^{(\alpha)}$ and $\hat{h}_i^{(\beta)}$—are linearly combined as below in order

might have minimum number of moments of order $\mathcal{O}(\varepsilon)$. We introduce

$$\left.\begin{aligned} \hat{\sigma}_{ij} &= \mathrm{x}^\circ_\alpha \hat{\sigma}^{(\alpha)}_{ij} + \mathrm{x}^\circ_\beta \hat{\sigma}^{(\beta)}_{ij}, & \Delta\hat{\sigma}_{ij} &= \kappa_1 \hat{\sigma}^{(\alpha)}_{ij} - \kappa_2 \hat{\sigma}^{(\beta)}_{ij}, \\ \hat{h}_i &= \frac{\mathrm{x}^\circ_\alpha}{\varkappa_\alpha} \hat{h}^{(\alpha)}_i + \frac{\mathrm{x}^\circ_\beta}{\varkappa_\beta} \hat{h}^{(\beta)}_i, & \Delta\hat{h}_i &= \kappa_3 \hat{h}^{(\alpha)}_i - \kappa_4 \hat{h}^{(\beta)}_i, \end{aligned}\right\} \tag{37}$$

where $\hat{\sigma}_{ij}$ and $\hat{h}_i$ are the (dimensionless) total stress and the (dimensionless) total reduced heat flux in the mixture, respectively, and

$$\left.\begin{aligned} \kappa_1 &= \varkappa_\beta(\mathrm{x}^\circ_\alpha\Omega_\alpha + \mathrm{x}^\circ_\beta\delta_3) + \varkappa_\alpha \mathrm{x}^\circ_\alpha \gamma_4, & \kappa_2 &= \varkappa_\alpha(\mathrm{x}^\circ_\beta\Omega_\beta + \mathrm{x}^\circ_\alpha\gamma_3) + \varkappa_\beta \mathrm{x}^\circ_\beta \delta_4, \\ \kappa_3 &= \left(\frac{2}{3}\mathrm{x}^\circ_\alpha\Omega_\alpha + \mathrm{x}^\circ_\beta\delta_5\right) + \mathrm{x}^\circ_\alpha\gamma_6, & \kappa_4 &= \left(\frac{2}{3}\mathrm{x}^\circ_\beta\Omega_\beta + \mathrm{x}^\circ_\alpha\gamma_5\right) + \mathrm{x}^\circ_\beta\delta_6, \end{aligned}\right\} \tag{38}$$

so that the leading orders of the total stress $\hat{\sigma}_{ij}$ and the total reduced heat flux $\hat{h}_i$ are one while the leading orders of $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ are two.

Thus, the minimum moments of order $\mathcal{O}(\varepsilon)$ are any one of the two diffusion velocities of the constituents, let us say $\hat{u}^{(\alpha)}_i$, the total stress $\hat{\sigma}_{ij}$ and the total reduced heat flux $\hat{h}_i$.

From Eqs. (37), one can obtain the expressions for the stresses and the reduced heat fluxes of the individual components in terms of the other variables. These will be needed in getting the minimum number of moments of order $\mathcal{O}(\varepsilon^2)$, and they read as follows.

$$\left.\begin{aligned} \hat{\sigma}^{(\alpha)}_{ij} &= \frac{\kappa_2\hat{\sigma}_{ij} + \mathrm{x}^\circ_\beta\Delta\hat{\sigma}_{ij}}{\mathrm{x}^\circ_\alpha\kappa_2 + \mathrm{x}^\circ_\beta\kappa_1}, & \hat{\sigma}^{(\beta)}_{ij} &= \frac{\kappa_1\hat{\sigma}_{ij} - \mathrm{x}^\circ_\alpha\Delta\hat{\sigma}_{ij}}{\mathrm{x}^\circ_\alpha\kappa_2 + \mathrm{x}^\circ_\beta\kappa_1}, \\ \hat{h}^{(\alpha)}_i &= \frac{\varkappa_\alpha(\kappa_4\varkappa_\beta\hat{h}_i + \mathrm{x}^\circ_\beta\Delta\hat{h}_i)}{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta + \mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha}, & \hat{h}^{(\beta)}_i &= \frac{\varkappa_\beta(\kappa_3\varkappa_\alpha\hat{h}_i - \mathrm{x}^\circ_\alpha\Delta\hat{h}_i)}{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta + \mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha}. \end{aligned}\right\} \tag{39}$$

B. Minimum number of moments of $\mathcal{O}(\varepsilon^2)$

We have established in Section III that the order $\mathcal{O}(\varepsilon^2)$ quantities are $\Delta\hat{T}$, $\Delta\hat{\sigma}_{ij}$, $\Delta\hat{h}_i$, $\hat{m}^{(\alpha)}_{ijk}$, $\hat{m}^{(\beta)}_{ijk}$, $\hat{R}^{(\alpha)}_{ij}$, $\hat{R}^{(\beta)}_{ij}$, $\hat{\Delta}_\alpha$ and $\hat{\Delta}_\beta$. Notice from the leading order contributions of $\Delta\hat{T}$, $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ (cf. Eqs. (B9), $(D8)_2$ and $(D9)_2$) that $\Delta\hat{T}$, $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ can neither be linearly combined among themselves nor with any other moments in order to produce a quantity of order higher than order $\mathcal{O}(\varepsilon^2)$. However, the other moments in the list—$\hat{m}^{(\alpha)}_{ijk}$, $\hat{m}^{(\beta)}_{ijk}$, $\hat{R}^{(\alpha)}_{ij}$, $\hat{R}^{(\beta)}_{ij}$, $\hat{\Delta}_\alpha$ and $\hat{\Delta}_\beta$—can be linearly combined to produce some quantities of order $\mathcal{O}(\varepsilon^3)$. To this end, let us first write down the leading order contributions of $\hat{m}^{(\alpha)}_{ijk}$, $\hat{m}^{(\beta)}_{ijk}$, $\hat{R}^{(\alpha)}_{ij}$, $\hat{R}^{(\beta)}_{ij}$, $\hat{\Delta}_\alpha$ and $\hat{\Delta}_\beta$ (by solving Eqs. (B10), (B11) and (B12) of appendix B and using Eqs. (39)), which

$$\left.\begin{aligned} \hat{m}^{(\alpha)}_{ijk|2} &= -c^{(\alpha)}_m \frac{\partial \hat{\sigma}_{\langle ij|1}}{\partial \hat{x}_{k\rangle}}, & \hat{m}^{(\beta)}_{ijk|2} &= -c^{(\beta)}_m \frac{\partial \hat{\sigma}_{\langle ij|1}}{\partial \hat{x}_{k\rangle}}, \\ \hat{R}^{(\alpha)}_{ij|2} &= -c^{(\alpha)}_R \frac{\partial \hat{h}_{\langle i|1}}{\partial \hat{x}_{j\rangle}}, & \hat{R}^{(\beta)}_{ij|2} &= -c^{(\beta)}_R \frac{\partial \hat{h}_{\langle i|1}}{\partial \hat{x}_{j\rangle}}, \\ \hat{\Delta}_{\alpha|2} &= -c^{(\alpha)}_\Delta \frac{\partial \hat{h}_{i|1}}{\partial \hat{x}_i}, & \hat{\Delta}_{\beta|2} &= -c^{(\beta)}_\Delta \frac{\partial \hat{h}_{i|1}}{\partial \hat{x}_i}. \end{aligned}\right\} \tag{40}$$

For better readability, the coefficients $c^{(\alpha)}_m$, $c^{(\beta)}_m$, $c^{(\alpha)}_R$, $c^{(\beta)}_R$, $c^{(\alpha)}_\Delta$ and $c^{(\beta)}_\Delta$ are given in appendix C. The quantities, $\hat{m}^{(\alpha)}_{ijk}$, $\hat{m}^{(\beta)}_{ijk}$, $\hat{R}^{(\alpha)}_{ij}$, $\hat{R}^{(\beta)}_{ij}$, $\hat{\Delta}_\alpha$ and $\hat{\Delta}_\beta$ are now linearly combined as below in order to have minimum number of moments of order $\mathcal{O}(\varepsilon^2)$. We introduce

$$\left.\begin{aligned} \hat{m}_{ijk} &= \frac{\mathrm{x}^\circ_\alpha}{\varkappa_\alpha}\hat{m}^{(\alpha)}_{ijk} + \frac{\mathrm{x}^\circ_\beta}{\varkappa_\beta}\hat{m}^{(\beta)}_{ijk}, & \Delta\hat{m}_{ijk} &= \kappa_5\hat{m}^{(\alpha)}_{ijk} - \kappa_6\hat{m}^{(\beta)}_{ijk}, \\ \hat{R}_{ij} &= \frac{\mathrm{x}^\circ_\alpha}{\varkappa^2_\alpha}\hat{R}^{(\alpha)}_{ij} + \frac{\mathrm{x}^\circ_\beta}{\varkappa^2_\beta}\hat{R}^{(\beta)}_{ij}, & \Delta\hat{R}_{ij} &= \kappa_7\hat{R}^{(\alpha)}_{ij} - \kappa_8\hat{R}^{(\beta)}_{ij}, \\ \hat{\Delta} &= \frac{\mathrm{x}^\circ_\alpha}{\varkappa^2_\alpha}\hat{\Delta}_\alpha + \frac{\mathrm{x}^\circ_\beta}{\varkappa^2_\beta}\hat{\Delta}_\beta, & \Delta\hat{\Delta} &= \kappa_9\hat{\Delta}_\alpha - \kappa_{10}\hat{\Delta}_\beta, \end{aligned}\right\} \tag{41}$$

where $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$ are the respective (dimensionless) total higher moments in the mixture, and

$$\left.\begin{aligned} \kappa_5 &= \kappa_1\left(\frac{3}{2}\mathrm{x}^\circ_\alpha\Omega_\alpha + \mathrm{x}^\circ_\beta\delta_7\right) + \kappa_2\mathrm{x}^\circ_\alpha\gamma_8, & \kappa_6 &= \kappa_2\left(\frac{3}{2}\mathrm{x}^\circ_\beta\Omega_\beta + \mathrm{x}^\circ_\alpha\gamma_7\right) + \kappa_1\mathrm{x}^\circ_\beta\delta_8, \\ \kappa_7 &= \kappa_3\left(\frac{7}{6}\mathrm{x}^\circ_\alpha\Omega_\alpha + \mathrm{x}^\circ_\beta\delta_9\right) + \kappa_4\mathrm{x}^\circ_\alpha\gamma_{10}, & \kappa_8 &= \kappa_4\left(\frac{7}{6}\mathrm{x}^\circ_\beta\Omega_\beta + \mathrm{x}^\circ_\alpha\gamma_9\right) + \kappa_3\mathrm{x}^\circ_\beta\delta_{10}, \\ \kappa_9 &= \kappa_3\left(\frac{2}{3}\mathrm{x}^\circ_\alpha\Omega_\alpha + \mathrm{x}^\circ_\beta\delta_{11}\right) + \kappa_4\mathrm{x}^\circ_\alpha\gamma_{12}, & \kappa_{10} &= \kappa_4\left(\frac{2}{3}\mathrm{x}^\circ_\beta\Omega_\beta + \mathrm{x}^\circ_\alpha\gamma_{11}\right) + \kappa_3\mathrm{x}^\circ_\beta\delta_{12}, \end{aligned}\right\} \tag{42}$$

so that the leading orders of $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$ are two while the leading orders of $\Delta\hat{m}_{ijk}$, $\Delta\hat{R}_{ij}$ and $\Delta\hat{\Delta}$ are three. Thus, the minimum moments of order $\mathcal{O}(\varepsilon^2)$ are $\Delta\hat{T}$, $\Delta\hat{\sigma}_{ij}$, $\Delta\hat{h}_i$, $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$. Notice, again, that the total higher order moments (m_{ijk}, R_{ij}, Δ) are scaled as

$$\hat{m}_{ijk} = \frac{m_{ijk}}{k\,n_\circ T_\circ v_\circ}, \quad \hat{R}_{ij} = \frac{R_{ij}}{k\,n_\circ T_\circ v_\circ^2} \quad \text{and} \quad \hat{\Delta} = \frac{\Delta}{k\,n_\circ T_\circ v_\circ^2}.$$

From Eqs. (41), one can obtain the expressions for the higher moments of the individual components in terms of the other variables. These will be needed later and they read as

follows.

$$\left.\begin{aligned}
\hat{m}_{ijk}^{(\alpha)} &= \frac{\varkappa_\alpha(\kappa_6\varkappa_\beta\hat{m}_{ijk} + \mathrm{x}_\beta^\circ\Delta\hat{m}_{ijk})}{\mathrm{x}_\alpha^\circ\kappa_6\varkappa_\beta + \mathrm{x}_\beta^\circ\kappa_5\varkappa_\alpha}, & \hat{m}_{ijk}^{(\beta)} &= \frac{\varkappa_\beta(\kappa_5\varkappa_\alpha\hat{m}_{ijk} - \mathrm{x}_\alpha^\circ\Delta\hat{m}_{ijk})}{\mathrm{x}_\alpha^\circ\kappa_6\varkappa_\beta + \mathrm{x}_\beta^\circ\kappa_5\varkappa_\alpha},\\
\hat{R}_{ij}^{(\alpha)} &= \frac{\varkappa_\alpha^2(\kappa_8\varkappa_\beta^2\hat{R}_{ij} + \mathrm{x}_\beta^\circ\Delta\hat{R}_{ij})}{\mathrm{x}_\alpha^\circ\kappa_8\varkappa_\beta^2 + \mathrm{x}_\beta^\circ\kappa_7\varkappa_\alpha^2}, & \hat{R}_{ij}^{(\beta)} &= \frac{\varkappa_\beta^2(\kappa_7\varkappa_\alpha^2\hat{R}_{ij} - \mathrm{x}_\alpha^\circ\Delta\hat{R}_{ij})}{\mathrm{x}_\alpha^\circ\kappa_8\varkappa_\beta^2 + \mathrm{x}_\beta^\circ\kappa_7\varkappa_\alpha^2},\\
\hat{\Delta}_\alpha &= \frac{\varkappa_\alpha^2(\kappa_{10}\varkappa_\beta^2\hat{\Delta} + \mathrm{x}_\beta^\circ\Delta\hat{\Delta})}{\mathrm{x}_\alpha^\circ\kappa_{10}\varkappa_\beta^2 + \mathrm{x}_\beta^\circ\kappa_9\varkappa_\alpha^2}, & \hat{\Delta}_\beta &= \frac{\varkappa_\beta^2(\kappa_9\varkappa_\alpha^2\hat{\Delta} - \mathrm{x}_\alpha^\circ\Delta\hat{\Delta})}{\mathrm{x}_\alpha^\circ\kappa_{10}\varkappa_\beta^2 + \mathrm{x}_\beta^\circ\kappa_9\varkappa_\alpha^2}.
\end{aligned}\right\} \tag{43}$$

VII. MOMENT EQUATIONS WITH λ^{th} ORDER ACCURACY

A. New system of equations

In the following, we shall write the conservation laws (Eqs. (6)–(10)) and Eqs. (18)–(30) in new variables $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$, $\Delta\hat{\sigma}_{ij}$, $\hat{h}_i$, $\Delta\hat{h}_i$, $\hat{m}_{ijk}$, $\Delta\hat{m}_{ijk}$, $\hat{R}_{ij}$, $\Delta\hat{R}_{ij}$, $\hat{\Delta}$, $\Delta\hat{\Delta}$ using Eqs. (37) and (41). It is emphasized, however, that this change of variables is required only for deriving the third order accurate equations, which we are interested in, and it may not be required to change all the variables for the derivation of zeroth, first and second order accurate equations. Additionally, we shall write each moment by assigning its magnitude in powers of ε ("in gray colour") in the new equations. These gray coloured ε's are included just for finding the terms of correct order while comparing the powers of ε on both sides (see below) and, of course, the value of gray coloured ε is essentially 1.

The conservation laws (Eqs. (6)–(10)) in new variables read

$$\varkappa_\alpha\left(\frac{\partial\hat{n}_\alpha}{\partial\hat{t}} + \frac{\partial\hat{v}_i}{\partial\hat{x}_i}\right) + \varepsilon\frac{\partial\hat{u}_i^{(\alpha)}}{\partial\hat{x}_i} = 0, \tag{44}$$

$$\frac{\partial}{\partial\hat{t}}\left(\mu_\alpha\mathrm{x}_\alpha^\circ\hat{n}_\alpha + \mu_\beta\mathrm{x}_\beta^\circ\hat{n}_\beta\right) + \left(\mu_\alpha\mathrm{x}_\alpha^\circ + \mu_\beta\mathrm{x}_\beta^\circ\right)\frac{\partial\hat{v}_i}{\partial\hat{x}_i} = 0, \tag{45}$$

$$\varkappa\frac{\partial\hat{v}_i}{\partial\hat{t}} + \varepsilon\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j} + \mathrm{x}_\alpha^\circ\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_i} + \mathrm{x}_\beta^\circ\frac{\partial\hat{n}_\beta}{\partial\hat{x}_i} + \frac{\partial\hat{T}}{\partial\hat{x}_i} = 0, \tag{46}$$

$$\frac{3}{2}\frac{\partial\hat{T}}{\partial\hat{t}} + \frac{\partial\hat{v}_i}{\partial\hat{x}_i} + \varepsilon\frac{\partial\hat{h}_i}{\partial\hat{x}_i} + \varsigma_1\varepsilon\frac{\partial\hat{u}_i^{(\alpha)}}{\partial\hat{x}_i} = 0. \tag{47}$$

Note that the mass balance equation for the β-constituent (7) is not included in this system as it can be obtained from Eqs. (44) and (45). The other equations in the new variables

read

$$\varkappa_\alpha \varepsilon \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{t}} + \varsigma_2 \varepsilon \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \varsigma_3 \varepsilon^2 \frac{\partial \Delta \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \mathrm{x}_\beta^\circ \varepsilon^2 \frac{\partial \Delta \hat{T}}{\partial \hat{x}_i}$$
$$= -\delta_1 \frac{1}{\varepsilon\, \Omega} \frac{\varkappa}{\varkappa_\beta^2} \left[\varepsilon \hat{u}_i^{(\alpha)} + \mathrm{x}_\beta^\circ \frac{\varepsilon\, \Omega}{\delta_1} \frac{\varkappa_\beta^2}{\varkappa^2} \left(\varkappa_\beta^2 \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} - \varkappa_\alpha^2 \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} - (\varkappa_\alpha^2 - \varkappa_\beta^2) \frac{\partial \hat{T}}{\partial \hat{x}_i} \right) \right], \tag{48}$$

$$\frac{3}{2} \varepsilon^2 \frac{\partial \Delta \hat{T}}{\partial \hat{t}} + \varsigma_4 \varepsilon \frac{\partial \hat{h}_i}{\partial \hat{x}_i} + \varsigma_5 \varepsilon^2 \frac{\partial \Delta \hat{h}_i}{\partial \hat{x}_i} + \varsigma_6 \varepsilon \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\, \Omega} \frac{\delta_2}{\varkappa_\alpha} \varepsilon^2 \Delta \hat{T}, \tag{49}$$

$$\varepsilon \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{t}} + \varepsilon^2 \frac{\partial \hat{m}_{ijk}}{\partial \hat{x}_k} + \frac{4}{5} \varepsilon \frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + 2\varsigma_1 \varepsilon \frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\, \Omega} \left[\varpi_1 \left(\varepsilon \hat{\sigma}_{ij} + 2\varepsilon\eta \frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}} \right) + \varpi_2 \varepsilon^2 \Delta \hat{\sigma}_{ij} \right], \tag{50}$$

$$\varepsilon^2 \frac{\partial \Delta \hat{\sigma}_{ij}}{\partial \hat{t}} + \varsigma_7 \varepsilon^2 \frac{\partial \hat{m}_{ijk}}{\partial \hat{x}_k} + \varsigma_8 \varepsilon^3 \frac{\partial \Delta \hat{m}_{ijk}}{\partial \hat{x}_k} + \frac{4}{5} \varsigma_9 \varepsilon \frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + \frac{4}{5} \varsigma_{10} \varepsilon^2 \frac{\partial \Delta \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + 2\varsigma_{11} \varepsilon \frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}}$$
$$= -\frac{1}{\varepsilon\, \Omega} \left[\varpi_3 \left(\varepsilon \hat{\sigma}_{ij} + 2\varepsilon\eta \frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}} \right) + \varpi_4 \varepsilon^2 \Delta \hat{\sigma}_{ij} \right], \tag{51}$$

$$\varepsilon \frac{\partial \hat{h}_i}{\partial \hat{t}} + \varsigma_{12} \varepsilon \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \varsigma_{13} \varepsilon^2 \frac{\partial \Delta \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \frac{1}{2} \varepsilon^2 \frac{\partial \hat{R}_{ij}}{\partial \hat{x}_j} + \frac{1}{6} \varepsilon^2 \frac{\partial \hat{\Delta}}{\partial \hat{x}_i} + \frac{5}{2} \varsigma_{14} \varepsilon^2 \frac{\partial \Delta \hat{T}}{\partial \hat{x}_i}$$
$$= -\frac{1}{\varepsilon\, \Omega} \left[\varpi_5 \left(\varepsilon \hat{h}_i + \varepsilon\kappa \frac{\partial \hat{T}}{\partial \hat{x}_i} \right) + \varpi_6 \varepsilon^2 \Delta \hat{h}_i \right], \tag{52}$$

$$\varepsilon^2 \frac{\partial \Delta \hat{h}_i}{\partial \hat{t}} + \varsigma_{15} \varepsilon \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \varsigma_{16} \varepsilon^2 \frac{\partial \Delta \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \frac{1}{2} \varsigma_{17} \varepsilon^2 \frac{\partial \hat{R}_{ij}}{\partial \hat{x}_j} + \frac{1}{2} \varsigma_{18} \varepsilon^3 \frac{\partial \Delta \hat{R}_{ij}}{\partial \hat{x}_j}$$
$$+ \frac{1}{6} \varsigma_{19} \varepsilon^2 \frac{\partial \hat{\Delta}}{\partial \hat{x}_i} + \frac{1}{6} \varsigma_{20} \varepsilon^3 \frac{\partial \Delta \hat{\Delta}}{\partial \hat{x}_i} + \frac{5}{2} \varsigma_{21} \varepsilon^2 \frac{\partial \Delta \hat{T}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\, \Omega} \left[\varpi_7 \left(\varepsilon \hat{h}_i + \varepsilon\kappa \frac{\partial \hat{T}}{\partial \hat{x}_i} \right) + \varpi_8 \varepsilon^2 \Delta \hat{h}_i \right]. \tag{53}$$

$$\varepsilon^2 \frac{\partial \hat{m}_{ijk}}{\partial \hat{t}} + \frac{3}{7} \varepsilon^2 \frac{\partial \hat{R}_{\langle ij}}{\partial \hat{x}_{k\rangle}} + 3\varsigma_{13} \varepsilon^2 \frac{\partial \Delta \hat{\sigma}_{\langle ij}}{\partial \hat{x}_{k\rangle}} = -\frac{1}{\varepsilon\, \Omega} \left[\varpi_9 \left(\varepsilon^2 \hat{m}_{ijk} + \varepsilon \zeta_m \varepsilon \frac{\partial \hat{\sigma}_{\langle ij}}{\partial \hat{x}_{k\rangle}} \right) + \varpi_{10} \varepsilon^3 \Delta \hat{m}_{ijk} \right], \tag{54}$$

$$\varepsilon^3\frac{\partial\Delta\hat{m}_{ijk}}{\partial\hat{t}}+\frac{3}{7}\varsigma_{22}\varepsilon^2\frac{\partial\hat{R}_{\langle ij}}{\partial\hat{x}_{k\rangle}}+\frac{3}{7}\varsigma_{23}\varepsilon^3\frac{\partial\Delta\hat{R}_{\langle ij}}{\partial\hat{x}_{k\rangle}}+3\varsigma_{24}\varepsilon^2\frac{\partial\Delta\hat{\sigma}_{\langle ij}}{\partial\hat{x}_{k\rangle}}$$
$$=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_{11}\left(\varepsilon^2\hat{m}_{ijk}+\varepsilon\zeta_m\varepsilon\frac{\partial\hat{\sigma}_{\langle ij}}{\partial\hat{x}_{k\rangle}}\right)+\varpi_{12}\varepsilon^3\Delta\hat{m}_{ijk}\right], \tag{55}$$

$$\varepsilon^2\frac{\partial\hat{R}_{ij}}{\partial\hat{t}}+2\varsigma_{25}\varepsilon^2\frac{\partial\hat{m}_{ijk}}{\partial\hat{x}_k}+2\varsigma_{26}\varepsilon^3\frac{\partial\Delta\hat{m}_{ijk}}{\partial\hat{x}_k}+\frac{28}{5}\varsigma_{27}\varepsilon^2\frac{\partial\Delta\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}$$
$$=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_{13}\left(\varepsilon^2\hat{R}_{ij}+\varepsilon\zeta_R\varepsilon\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)+\varpi_{14}\varepsilon^3\Delta\hat{R}_{ij}\right], \tag{56}$$

$$\varepsilon^3\frac{\partial\Delta\hat{R}_{ij}}{\partial\hat{t}}+2\varsigma_{28}\varepsilon^2\frac{\partial\hat{m}_{ijk}}{\partial\hat{x}_k}+2\varsigma_{29}\varepsilon^3\frac{\partial\Delta\hat{m}_{ijk}}{\partial\hat{x}_k}+\frac{28}{5}\varsigma_{30}\varepsilon^2\frac{\partial\Delta\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}$$
$$=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_{15}\left(\varepsilon^2\hat{R}_{ij}+\varepsilon\zeta_R\varepsilon\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)+\varpi_{16}\varepsilon^3\Delta\hat{R}_{ij}\right], \tag{57}$$

$$\varepsilon^2\frac{\partial\hat{\Delta}}{\partial\hat{t}}+8\varsigma_{27}\varepsilon^2\frac{\partial\Delta\hat{h}_i}{\partial\hat{x}_i}=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_{17}\left(\varepsilon^2\hat{\Delta}+\varepsilon\zeta_\Delta\varepsilon\frac{\partial\hat{h}_i}{\partial\hat{x}_i}\right)+\varpi_{18}\varepsilon^3\Delta\hat{\Delta}\right], \tag{58}$$

$$\varepsilon^3\frac{\partial\Delta\hat{\Delta}}{\partial\hat{t}}+8\varsigma_{31}\varepsilon^2\frac{\partial\Delta\hat{h}_i}{\partial\hat{x}_i}=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_{19}\left(\varepsilon^2\hat{\Delta}+\varepsilon\zeta_\Delta\varepsilon\frac{\partial\hat{h}_i}{\partial\hat{x}_i}\right)+\varpi_{20}\varepsilon^3\Delta\hat{\Delta}\right], \tag{59}$$

where

$$\eta=\frac{\Omega}{\varpi_1}=\mathrm{x}_\alpha^\circ\eta_\alpha+\mathrm{x}_\beta^\circ\eta_\beta\quad\text{and}\quad\kappa=\frac{5}{2}\frac{\Omega}{\varpi_5}\left(\frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha^2}+\frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta^2}\right)=\frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha}\kappa_\alpha+\frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta}\kappa_\beta \tag{60}$$

are the dimensionless viscosity and the dimensionless heat conductivity, respectively, of the mixture, and all other coefficients are given in appendix C. The balance equation for diffusion velocity of the β-constituent (19) is also not included in the above system, since it can be obtained from the balance equation for diffusion velocity of the α-constituent (18).

a. λ^{th} order accuracy: Clearly, the conservation laws (44)–(47) do not form a closed set of equations for $\hat{n}_\alpha$, $\hat{n}_\beta$, $\hat{v}_i$, $\hat{T}$ because they contain the additional variables $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$, $\hat{h}_i$. We shall speak of a theory with λ^{th} order accuracy, when $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$ are accurately known up to order $\mathcal{O}(\varepsilon^\lambda)$.

B. Zeroth order accuracy: Euler equations

The equations with zeroth order accuracy result by setting the first order quantities to zero, i.e., by ignoring the terms with the factor ε in the conservation laws (44)–(47). This

yields the (linearized) Euler equations for a binary mixture of gases α and β:

$$\left.\begin{aligned} \frac{\partial \hat{n}_\alpha}{\partial \hat{t}} + \frac{\partial \hat{v}_i}{\partial \hat{x}_i} &= 0, \\ \frac{\partial}{\partial \hat{t}}\left(\mu_\alpha \mathrm{x}_\alpha^\circ \hat{n}_\alpha + \mu_\beta \mathrm{x}_\beta^\circ \hat{n}_\beta\right) + \left(\mu_\alpha \mathrm{x}_\alpha^\circ + \mu_\beta \mathrm{x}_\beta^\circ\right)\frac{\partial \hat{v}_i}{\partial \hat{x}_i} &= 0, \\ \varkappa \frac{\partial \hat{v}_i}{\partial \hat{t}} + \mathrm{x}_\alpha^\circ \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} + \mathrm{x}_\beta^\circ \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} + \frac{\partial \hat{T}}{\partial \hat{x}_i} &= 0, \\ \frac{3}{2}\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial \hat{v}_i}{\partial \hat{x}_i} &= 0. \end{aligned}\right\} \tag{61}$$

C. First order accuracy: Fick, Navier–Stokes, and Fourier equations

For first order accuracy, one needs to include all the terms with factors ε^0 and ε^1. That means all the terms in the conservation laws (44)–(47) are retained, and therefore the conservation laws at this order (on setting gray coloured ε to 1) read

$$\left.\begin{aligned} \varkappa_\alpha\left(\frac{\partial \hat{n}_\alpha}{\partial \hat{t}} + \frac{\partial \hat{v}_i}{\partial \hat{x}_i}\right) + \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i} &= 0, \\ \frac{\partial}{\partial \hat{t}}\left(\mu_\alpha \mathrm{x}_\alpha^\circ \hat{n}_\alpha + \mu_\beta \mathrm{x}_\beta^\circ \hat{n}_\beta\right) + \left(\mu_\alpha \mathrm{x}_\alpha^\circ + \mu_\beta \mathrm{x}_\beta^\circ\right)\frac{\partial \hat{v}_i}{\partial \hat{x}_i} &= 0, \\ \varkappa \frac{\partial \hat{v}_i}{\partial \hat{t}} + \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \mathrm{x}_\alpha^\circ \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} + \mathrm{x}_\beta^\circ \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} + \frac{\partial \hat{T}}{\partial \hat{x}_i} &= 0, \\ \frac{3}{2}\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial \hat{v}_i}{\partial \hat{x}_i} + \frac{\partial \hat{h}_i}{\partial \hat{x}_i} + \varsigma_1 \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i} &= 0, \end{aligned}\right\} \tag{62}$$

where we need to find $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$ accurately up to first order, i.e., to their leading orders. For the leading orders of these quantities, only the terms up to order $\mathcal{O}(\varepsilon^0)$ in the balance equations for these quantities (Eqs. (48), (50) and (52)) need to be considered and, obviously, there are no terms of order $\mathcal{O}(\varepsilon^0)$ on the left-hand sides of Eqs. (48), (50) and (52). Thus, we readily obtain the first order accurate $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$, which—on setting gray coloured ε to 1—are the laws of Fick, Navier–Stokes, and Fourier:

$$\left.\begin{aligned} \hat{u}_i^{(\alpha)} &= -\mathrm{x}_\beta^\circ \frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2 \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} - \varkappa_\alpha^2 \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_i}\right), \\ \hat{\sigma}_{ij} &= -2\varepsilon\eta \frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}, \\ \hat{h}_i &= -\varepsilon\kappa \frac{\partial \hat{T}}{\partial \hat{x}_i}. \end{aligned}\right\} \tag{63}$$

Equations (62) along with constitutive relations (63) form the system of (linearized) Fick, Navier–Stokes, and Fourier equations for a binary mixture of gases α and β, where η and κ are the dimensionless viscosity and dimensionless heat conductivity, respectively, and they are given by Eqs. (60).

We have also compared the transport coefficients obtained here with those obtained through the classical Chapman–Enskog expansion method in Ref. 8 and found that the dimensionless viscosity obtained here match with that obtained via the classical Chapman–Enskog expansion method in Ref. 8. In order to have more insight into the other transport coefficients, let us compare the expression for diffusion velocity and (reduced) heat flux obtained here with those obtained through the classical Chapman–Enskog expansion method in Ref. 8. The diffusion velocity of component α given by Eq. $(63)_1$ is the linear-dimensionless form of the diffusion velocity of component α[8,10]

$$u_i^{(\alpha)} = \frac{n^2}{n_\alpha \rho} m_\beta D_{\alpha\beta}\, d_i^{(\beta)} - \underline{\frac{1}{\rho_\alpha} D_\alpha^T \frac{\partial \ln T}{\partial x_i}} \tag{64}$$

when the underlined term in Eq. (64) vanishes, and the (reduced) heat flux given by Eq. $(63)_3$ is the linear-dimensionless form of the total heat flux[8]

$$q_i = -\left(\lambda^* + \underline{\frac{k}{n}\frac{\rho^2}{\rho_\alpha \rho_\beta}\frac{1}{m_\alpha m_\beta}\frac{D_\alpha^T D'_\alpha}{D_{\alpha\beta}}}\right)\frac{\partial T}{\partial x_i} + \frac{5}{2}kT\left(n_\alpha u_i^{(\alpha)} + n_\beta u_i^{(\beta)}\right) - \underline{nkT\frac{\rho}{\rho_\alpha \rho_\beta} D'_\alpha\, d_i^{(\alpha)}} \tag{65}$$

when the underlined terms in Eq. (65) vanish. Here, $D_{\alpha\beta}$, D_α^T and D'_α are the diffusion, thermal diffusion and diffusion-thermal coefficients, respectively;

$$d_i^{(\gamma)} = \frac{\partial}{\partial x_i}\left(\frac{n_\gamma}{n}\right) + \left(\frac{n_\gamma}{n} - \frac{\rho_\gamma}{\rho}\right)\frac{\partial \ln p}{\partial x_i} \tag{66}$$

is the so-called generalized diffusion force[10] of the constituent $\gamma \in \{\alpha, \beta\}$; and λ^* is the thermal conductivity of the mixture.

Comparing the dimensionless form of Eq. (64) with Eq. $(63)_1$, the (ground state) diffusion coefficient turns out to be

$$D^\circ_{\alpha\beta} = \frac{\varepsilon\,\Omega\, L\sqrt{\theta^\circ_\alpha}}{\delta_1} = \frac{3}{2\sqrt{2}}\frac{\varepsilon\,\Omega\, L}{a_1}\frac{1}{\sqrt{\mu_\alpha \mu_\beta}}\sqrt{\frac{k\,T_\circ}{m_\alpha + m_\beta}} = D^\circ_{\beta\alpha} \tag{67}$$

and the thermal diffusion coefficient D_α^T vanishes at this order both in our computation as well as in Ref. 8. Similarly, on comparing the dimensionless form of Eq. (65) with Eq. $(63)_3$, it turns out that the diffusion-thermal coefficient D'_α also vanishes at this order and the thermal conductivity of the binary gas mixture is $\lambda^* = (\varepsilon L k n_\circ v_\circ)\,\kappa$. Notice that the zero thermal diffusion coefficient—at first order in the Chapman–Enskog expansion—in binary gas mixtures of Maxwell molecules is also attributed to Maxwell interaction potential, see Eqs. (8.142), (8.147) and (8.155) of Ref. 10. Thus, diffusion in the binary gas mixtures of Maxwell molecules occurs due to molar concentration gradients and pressure gradient but not explicitly due to the temperature gradient at first order, even though the temperature gradient does appear through the pressure gradient term. This means that the cross-effects of thermal diffusion and diffusion-thermal are not present in binary gas mixtures of Maxwell molecules at first order. Nevertheless, our results do satisfy the Onsager's reciprocity relations:[53,54] $D^\circ_{\alpha\beta} = D^\circ_{\beta\alpha}$ and $D_\alpha^T = D'_\alpha = 0$.

D. Second order accuracy: 17 moment equations

At this order, we need to find $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$, appearing in the conservation laws, with second order accuracy. Therefore, one needs to consider all terms having factors ε^0 and ε^1 in the balance equations of these quantities (i.e., in Eqs. (48), (50) and (52)), we have (on setting gray coloured ε to 1)

$$\varkappa_\alpha \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{t}} + \varsigma_2 \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} = -\delta_1 \frac{1}{\varepsilon\,\Omega} \frac{\varkappa}{\varkappa_\beta^2} \left[\hat{u}_i^{(\alpha)} + \mathrm{x}_\beta^\circ \frac{\varepsilon\,\Omega}{\delta_1} \frac{\varkappa_\beta^2}{\varkappa^2} \left(\varkappa_\beta^2 \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} - \varkappa_\alpha^2 \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} - (\varkappa_\alpha^2 - \varkappa_\beta^2) \frac{\partial \hat{T}}{\partial \hat{x}_i} \right) \right], \tag{68}$$

$$\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{t}} + \frac{4}{5} \frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + 2\varsigma_1 \frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega} \left[\varpi_1 \left(\hat{\sigma}_{ij} + 2\varepsilon\eta \frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}} \right) + \varpi_2 \Delta\hat{\sigma}_{ij} \right], \tag{69}$$

$$\frac{\partial \hat{h}_i}{\partial \hat{t}} + \varsigma_{12} \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} = -\frac{1}{\varepsilon\,\Omega} \left[\varpi_5 \left(\hat{h}_i + \varepsilon\kappa \frac{\partial \hat{T}}{\partial \hat{x}_i} \right) + \varpi_6 \Delta\hat{h}_i \right], \tag{70}$$

where $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ are needed to be second order accurate. The second order accurate $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ follow from their respective balance equations (Eqs. (51) and (53)) on considering terms up to order $\mathcal{O}(\varepsilon)$, we have (on setting gray coloured ε to 1)

$$\Delta\hat{\sigma}_{ij} \approx \Delta\hat{\sigma}_{ij}^{(2)} = -\frac{\varpi_3}{\varpi_4} \left(\hat{\sigma}_{ij} + 2\varepsilon\eta \frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}} \right) - \frac{\varepsilon\,\Omega}{\varpi_4} \left(\frac{4}{5}\varsigma_9 \frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + 2\varsigma_{11} \frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} \right), \tag{71}$$

$$\Delta\hat{h}_i \approx \Delta\hat{h}_i^{(2)} = -\frac{\varpi_7}{\varpi_8} \left(\hat{h}_i + \varepsilon\kappa \frac{\partial \hat{T}}{\partial \hat{x}_i} \right) - \frac{\varepsilon\,\Omega}{\varpi_8} \varsigma_{15} \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j}. \tag{72}$$

where the superscript '(2)' denotes the second order accurate contributions. Thus, we deduce that even though the temperature difference $\Delta\hat{T}$ is a quantity of leading order two, it is not a relevant quantity for a second order accurate theory.

In this way, the system of (linearized) second order accurate equations consists of the conservation laws (62) and the governing equations for $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$ (Eqs. (68)–(70))—a total of 17 equations in three dimensions (3D)—and it is closed with the second order accurate contributions of $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$, given by Eqs. (71) and (72). The (linearized)

second order accurate equations in the closed form read

$$\left.\begin{aligned}
\varkappa_\alpha\left(\frac{\partial \hat{n}_\alpha}{\partial \hat{t}}+\frac{\partial \hat{v}_i}{\partial \hat{x}_i}\right)+\frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i}&=0,\\
\frac{\partial}{\partial \hat{t}}\left(\mu_\alpha \mathrm{x}_\alpha^\circ \hat{n}_\alpha+\mu_\beta \mathrm{x}_\beta^\circ \hat{n}_\beta\right)+\left(\mu_\alpha \mathrm{x}_\alpha^\circ+\mu_\beta \mathrm{x}_\beta^\circ\right)\frac{\partial \hat{v}_i}{\partial \hat{x}_i}&=0,\\
\varkappa\frac{\partial \hat{v}_i}{\partial \hat{t}}+\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j}+\mathrm{x}_\alpha^\circ\frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i}+\mathrm{x}_\beta^\circ\frac{\partial \hat{n}_\beta}{\partial \hat{x}_i}+\frac{\partial \hat{T}}{\partial \hat{x}_i}&=0,\\
\frac{3}{2}\frac{\partial \hat{T}}{\partial \hat{t}}+\frac{\partial \hat{v}_i}{\partial \hat{x}_i}+\frac{\partial \hat{h}_i}{\partial \hat{x}_i}+\varsigma_1\frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{x}_i}&=0,
\end{aligned}\right\} \tag{73}$$

$$\varkappa_\alpha\frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{t}}+\mathfrak{a}_0\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j}=-\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_1\left[\hat{u}_i^{(\alpha)}+\mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i}-\varkappa_\alpha^2\frac{\partial \hat{n}_\beta}{\partial \hat{x}_i}-(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_i}\right)\right], \tag{74}$$

$$\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{t}}+\mathfrak{a}_2\frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}}+\mathfrak{a}_3\frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}}=-\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_4\left(\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right), \tag{75}$$

$$\frac{\partial \hat{h}_i}{\partial \hat{t}}+\mathfrak{a}_5\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j}=-\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_6\left(\hat{h}_i+\varepsilon\kappa\frac{\partial \hat{T}}{\partial \hat{x}_i}\right). \tag{76}$$

The coefficients $\mathfrak{a}_0, \mathfrak{a}_2, \ldots, \mathfrak{a}_6$ are given in appendix C. Notice that even the second order accurate equations (Eqs. (73)–(76)) cannot explain the cross-effects of thermal diffusion and diffusion-thermal since there is still no temperature gradient term in the governing equation for $\hat{u}_i^{(\alpha)}$ and no pressure or number density gradient terms in the governing equations for $\hat{h}_i$.

E. Third order accuracy: regularized moment equations

1. *Intermediate result: 25 equations*

At this order, we need to find $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$, appearing in the conservation laws, with third order accuracy. Therefore, one need to consider all terms having factors ε^0, ε^1 and ε^2 in the balance equations of these quantities (i.e., in Eqs. (48), (50) and (52)), we get (on setting gray coloured ε to 1)

$$\begin{aligned}
&\varkappa_\alpha\frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{t}}+\varsigma_2\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j}+\varsigma_3\frac{\partial \Delta\hat{\sigma}_{ij}}{\partial \hat{x}_j}+\mathrm{x}_\beta^\circ\frac{\partial \Delta\hat{T}}{\partial \hat{x}_i}\\
&=-\delta_1\frac{1}{\varepsilon\,\Omega}\frac{\varkappa}{\varkappa_\beta^2}\left[\hat{u}_i^{(\alpha)}+\mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i}-\varkappa_\alpha^2\frac{\partial \hat{n}_\beta}{\partial \hat{x}_i}-(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_i}\right)\right],
\end{aligned} \tag{77}$$

$$\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{t}} + \frac{\partial \hat{m}_{ijk}}{\partial \hat{x}_k} + \frac{4}{5}\frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + 2\varsigma_1 \frac{\partial \hat{u}^{(\alpha)}_{\langle i}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left[\varpi_1\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right) + \varpi_2 \Delta\hat{\sigma}_{ij}\right], \tag{78}$$

$$\frac{\partial \hat{h}_i}{\partial \hat{t}} + \varsigma_{12}\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \varsigma_{13}\frac{\partial \Delta\hat{\sigma}_{ij}}{\partial \hat{x}_j} + \frac{1}{2}\frac{\partial \hat{R}_{ij}}{\partial \hat{x}_j} + \frac{1}{6}\frac{\partial \hat{\Delta}}{\partial \hat{x}_i} + \frac{5}{2}\varsigma_{14}\frac{\partial \Delta\hat{T}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\,\Omega}\left[\varpi_5\left(\hat{h}_i + \varepsilon\kappa\frac{\partial \hat{T}}{\partial \hat{x}_i}\right) + \varpi_6\Delta\hat{h}_i\right]. \tag{79}$$

Now, we have the additional variables $\Delta\hat{T}$, $\Delta\hat{\sigma}_{ij}$, $\Delta\hat{h}_i$, $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$ in the system. The variables $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ not only appear on the left-hand sides of Eqs. (77)–(79) where only their leading order contributions are required but also on the right-hand sides of Eqs. (78) and (79) where they are required up to order $\mathcal{O}(\varepsilon^3)$. Therefore, we need to include the terms up to order $\mathcal{O}(\varepsilon^2)$ in the balance equations for them (Eqs. (51) and (53)), which gives (on setting gray coloured ε to 1)

$$\frac{\partial \Delta\hat{\sigma}_{ij}}{\partial \hat{t}} + \varsigma_7\frac{\partial \hat{m}_{ijk}}{\partial \hat{x}_k} + \frac{4}{5}\varsigma_9\frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + \frac{4}{5}\varsigma_{10}\frac{\partial \Delta\hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + 2\varsigma_{11}\frac{\partial \hat{u}^{(\alpha)}_{\langle i}}{\partial \hat{x}_{j\rangle}} = -\frac{1}{\varepsilon\,\Omega}\left[\varpi_3\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right) + \varpi_4\Delta\hat{\sigma}_{ij}\right], \tag{80}$$

$$\frac{\partial \Delta\hat{h}_i}{\partial \hat{t}} + \varsigma_{15}\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \varsigma_{16}\frac{\partial \Delta\hat{\sigma}_{ij}}{\partial \hat{x}_j} + \frac{1}{2}\varsigma_{17}\frac{\partial \hat{R}_{ij}}{\partial \hat{x}_j} + \frac{1}{6}\varsigma_{19}\frac{\partial \hat{\Delta}}{\partial \hat{x}_i} + \frac{5}{2}\varsigma_{21}\frac{\partial \Delta\hat{T}}{\partial \hat{x}_i} = -\frac{1}{\varepsilon\,\Omega}\left[\varpi_7\left(\hat{h}_i + \varepsilon\kappa\frac{\partial \hat{T}}{\partial \hat{x}_i}\right) + \varpi_8\Delta\hat{h}_i\right]. \tag{81}$$

Fortunately, all other additional variables—$\Delta\hat{T}$, $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$—appear only on the left-hand sides of Eqs. (77)–(81). Therefore, for the third order accurate $\hat{u}^{(\alpha)}_i$, $\hat{\sigma}_{ij}$, $\hat{h}_i$, $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$, only the second order accurate contributions of $\Delta\hat{T}$, $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$ are needed and these follow from their respective balance equations (Eqs. (49), (54), (56), (58), respectively) by considering only the terms up to order $\mathcal{O}(\varepsilon)$, we have (on setting gray coloured ε to 1)

$$\left.\begin{aligned} \Delta\hat{T} &= -\varepsilon\,\Omega\frac{\varkappa_\alpha}{\delta_2}\left(\varsigma_4\frac{\partial \hat{h}_i}{\partial \hat{x}_i} + \varsigma_6\frac{\partial \hat{u}^{(\alpha)}_i}{\partial \hat{x}_i}\right), \\ \hat{m}_{ijk} &= -\varepsilon\,\zeta_m\frac{\partial \hat{\sigma}_{\langle ij}}{\partial \hat{x}_{k\rangle}}, \quad \hat{R}_{ij} = -\varepsilon\,\zeta_R\frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}}, \quad \hat{\Delta} = -\varepsilon\,\zeta_\Delta\frac{\partial \hat{h}_i}{\partial \hat{x}_i}. \end{aligned}\right\} \tag{82}$$

Thus, the system of third order accurate equations consists of the conservation laws (62) and the governing equations for $\hat{u}^{(\alpha)}_i$, $\hat{\sigma}_{ij}$, $\hat{h}_i$, $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ (Eqs. (77)–(81))—a total of 25 equations in 3D—and the system is closed with the second order accurate contributions of $\Delta\hat{T}$, $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$, given by Eqs. (82).

2 *Further reduction*

As one can notice, Eqs. (80) and (81) have been included in the system of third order accurate equations just because $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ are present on the right-hand sides of Eqs. (78) and (79). Nevertheless, the explicit *third* order accurate expressions for $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ can be obtained by using ideas somewhat similar to the Chapman–Enskog expansion, also used in[28], so that we shall only have 17 equations in 3D and the third order accurate values of $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ can be included in the closures.

For finding the third order accurate $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$, it suffices to consider their second order accurate contributions on the left-hand sides of Eqs. (80) and (81). In other words, Eqs. (80) and (81)—on setting gray coloured ε to 1—can be rewritten as

$$\frac{\partial\Delta\hat{\sigma}_{ij}^{(2)}}{\partial\hat{t}}+\varsigma_7\frac{\partial\hat{m}_{ijk}}{\partial\hat{x}_k}+\frac{4}{5}\varsigma_9\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}+\frac{4}{5}\varsigma_{10}\frac{\partial\Delta\hat{h}_{\langle i}^{(2)}}{\partial\hat{x}_{j\rangle}}+2\varsigma_{11}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_3\left(\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)+\varpi_4\Delta\hat{\sigma}_{ij}\right], \tag{83}$$

$$\frac{\partial\Delta\hat{h}_i^{(2)}}{\partial\hat{t}}+\varsigma_{15}\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j}+\varsigma_{16}\frac{\partial\Delta\hat{\sigma}_{ij}^{(2)}}{\partial\hat{x}_j}+\frac{1}{2}\varsigma_{17}\frac{\partial\hat{R}_{ij}}{\partial\hat{x}_j}+\frac{1}{6}\varsigma_{19}\frac{\partial\hat{\Delta}}{\partial\hat{x}_i}+\frac{5}{2}\varsigma_{21}\frac{\partial\Delta\hat{T}}{\partial\hat{x}_i}=-\frac{1}{\varepsilon\,\Omega}\left[\varpi_7\left(\hat{h}_i+\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}\right)+\varpi_8\Delta\hat{h}_i\right]. \tag{84}$$

From Eqs. (71) and (72), we have

$$\left.\begin{aligned}\frac{\partial\Delta\hat{\sigma}_{ij}^{(2)}}{\partial\hat{t}}&=-\frac{\varpi_3}{\varpi_4}\frac{\partial}{\partial\hat{t}}\underbrace{\left(\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)}-\frac{\varepsilon\,\Omega}{\varpi_4}\left(\frac{4}{5}\varsigma_9\underline{\frac{\partial}{\partial\hat{x}_{j\rangle}}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{t}}}+2\varsigma_{11}\underline{\frac{\partial}{\partial\hat{x}_{j\rangle}}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{t}}}\right),\\ \frac{\partial\Delta\hat{h}_i^{(2)}}{\partial\hat{t}}&=-\frac{\varpi_7}{\varpi_8}\frac{\partial}{\partial\hat{t}}\underbrace{\left(\hat{h}_i+\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}\right)}-\frac{\varepsilon\,\Omega}{\varpi_8}\varsigma_{15}\underline{\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{t}}}.\end{aligned}\right\} \tag{85}$$

As we want to evaluate the time derivatives of the second order accurate $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$, it is natural to use the second order accurate balance equations for $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$ (Eqs. (68), (75) and (76)) for replacing the time derivatives in the underlined terms in Eqs. (85). Moreover, the underbraced terms in Eqs. (85) are order $\mathcal{O}(\varepsilon^2)$ contributions to the total stress and the total reduced heat flux and it suffices to use only the precise values of order $\mathcal{O}(\varepsilon^2)$ contributions of these quantities in Eqs. (85). The precise values of order $\mathcal{O}(\varepsilon^2)$ contributions to the total stress and the total reduced heat flux can be obtained by performing Chapman–Enskog like expansion either on the second order accurate balance equations (Eqs. (68), (75) and (76)) or on the full system of moment equations (Eqs. (48)–(59)) and we get

(cf. Eqs. (D12) and (D14))

$$\left.\begin{aligned}\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}&=-\varepsilon^2\frac{\partial^2}{\partial\hat{x}_{\langle i}\hat{x}_{j\rangle}}\big(\mathfrak{b}_1\hat{n}_\alpha+\mathfrak{b}_2\hat{n}_\beta+\mathfrak{b}_3\hat{T}\big)+\mathcal{O}(\varepsilon^3),\\ \hat{h}_i+\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}&=-\varepsilon^2\frac{\Omega}{\mathfrak{a}_6}\left(\frac{2}{3}\kappa\frac{\partial^2\hat{v}_j}{\partial\hat{x}_i\partial\hat{x}_j}-2\eta\mathfrak{a}_5\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)+\mathcal{O}(\varepsilon^3).\end{aligned}\right\}\tag{86}$$

The values of the coefficients $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ are given in appendix D. For the second order accurate underbraced terms in Eqs. (85), we can use $\hat{\sigma}_{ij}\approx-2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}$ in the right-hand side of Eq. $(86)_2$ and it will not affect the accuracy. Thus, we have

$$\left.\begin{aligned}\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}&\approx-\varepsilon^2\frac{\partial^2}{\partial\hat{x}_{\langle i}\hat{x}_{j\rangle}}\big(\mathfrak{b}_1\hat{n}_\alpha+\mathfrak{b}_2\hat{n}_\beta+\mathfrak{b}_3\hat{T}\big),\\ \hat{h}_i+\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}&\approx-\frac{\Omega}{\mathfrak{a}_6}\left(\frac{2}{3}\kappa\,\varepsilon^2\frac{\partial^2\hat{v}_j}{\partial\hat{x}_i\partial\hat{x}_j}+\mathfrak{a}_5\varepsilon\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j}\right).\end{aligned}\right\}\tag{87}$$

Now, we apply the time derivative and immediately replace the time derivative of the total stress with its second order accurate balance equation (69) and the time derivatives of number densities, velocity and temperature using the conservation laws with $\hat{u}_i^{(\alpha)}=\hat{u}_i^{(\beta)}=\hat{\sigma}_{ij}=\hat{h}_i=0$ (i.e., using Euler equations (61)) to get

$$\left.\begin{aligned}\frac{\partial}{\partial\hat{t}}\left(\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)&\approx\underline{\left(\mathfrak{b}_1+\mathfrak{b}_2+\frac{2}{3}\mathfrak{b}_3\right)\varepsilon^2\frac{\partial^2}{\partial\hat{x}_{\langle i}\hat{x}_{j\rangle}}\frac{\partial\hat{v}_k}{\partial\hat{x}_k}},\\ \frac{\partial}{\partial\hat{t}}\left(\hat{h}_i+\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}\right)&\approx\underline{\frac{2}{3}\frac{\Omega}{\mathfrak{a}_6}\frac{\kappa}{\varkappa}\varepsilon^2\frac{\partial^2}{\partial\hat{x}_i\partial\hat{x}_j}\frac{\partial}{\partial\hat{x}_j}\big(\mathrm{x}_\alpha^\circ\hat{n}_\alpha+\mathrm{x}_\beta^\circ\hat{n}_\beta+\hat{T}\big)}\\ &\quad+\frac{\Omega}{\mathfrak{a}_6}\mathfrak{a}_5\varepsilon\frac{\partial}{\partial\hat{x}_j}\left[\mathfrak{a}_2\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}+\mathfrak{a}_3\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}+\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_4\left(\hat{\sigma}_{ij}+2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)\right].\end{aligned}\right\}\tag{88}$$

The temperature gradient in Eq. $(88)_2$ is replaced by the reduced heat flux by using $\hat{h}_i\approx-\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}$; again, this change will not affect the accuracy. However, the elimination of gradients of the number densities requires the following argument. Similar to above, without affecting the accuracy, we use

$$\hat{u}_i^{(\alpha)}\approx-\mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_i}-\varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_i}-(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_i}\right)\quad\text{and}\quad\hat{h}_i\approx-\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}$$

in order to get

$$-\varepsilon\frac{1}{\mathrm{x}_\beta^\circ}\frac{\delta_1}{\Omega}\frac{\varkappa^2}{\varkappa_\beta^2}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}-\varepsilon(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{1}{\kappa}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}\approx\varepsilon^2\left(\varkappa_\beta^2\frac{\partial^2\hat{n}_\alpha}{\partial\hat{x}_{\langle i}\partial\hat{x}_{j\rangle}}-\varkappa_\alpha^2\frac{\partial^2\hat{n}_\beta}{\partial\hat{x}_{\langle i}\partial\hat{x}_{j\rangle}}\right).\tag{89}$$

Moreover, we again use $\hat{h}_i \approx -\varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}$ in Eq. $(87)_1$ to obtain

$$\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right) \approx -\varepsilon^2\frac{\partial^2}{\partial\hat{x}_{\langle i}\hat{x}_{j\rangle}}\left(\mathfrak{b}_1\hat{n}_\alpha + \mathfrak{b}_2\hat{n}_\beta\right) + \varepsilon\frac{\mathfrak{b}_3}{\kappa}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}, \tag{90}$$

On solving Eq. (89) with Eq. (90), one obtains

$$\begin{aligned}\varepsilon^2\frac{\partial^2\hat{n}_\alpha}{\partial\hat{x}_{\langle i}\partial\hat{x}_{j\rangle}} \approx \frac{1}{(\mathfrak{b}_1\varkappa_\alpha^2 + \mathfrak{b}_2\varkappa_\beta^2)}\Bigg[&-\varepsilon\frac{1}{\mathrm{x}_\beta^\circ}\frac{\delta_1}{\Omega}\frac{\varkappa^2}{\varkappa_\beta^2}\mathfrak{b}_2\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}} + \varepsilon\left\{\mathfrak{b}_3\varkappa_\alpha^2 - \mathfrak{b}_2(\varkappa_\alpha^2 - \varkappa_\beta^2)\right\}\frac{1}{\kappa}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}\\ &- \varkappa_\alpha^2\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)\Bigg],\end{aligned} \tag{91}$$

$$\begin{aligned}\varepsilon^2\frac{\partial^2\hat{n}_\beta}{\partial\hat{x}_{\langle i}\partial\hat{x}_{j\rangle}} \approx \frac{1}{(\mathfrak{b}_1\varkappa_\alpha^2 + \mathfrak{b}_2\varkappa_\beta^2)}\Bigg[&\varepsilon\frac{1}{\mathrm{x}_\beta^\circ}\frac{\delta_1}{\Omega}\frac{\varkappa^2}{\varkappa_\beta^2}\mathfrak{b}_1\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}} + \varepsilon\left\{\mathfrak{b}_1(\varkappa_\alpha^2 - \varkappa_\beta^2) + \mathfrak{b}_3\varkappa_\beta^2\right\}\frac{1}{\kappa}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}\\ &- \varkappa_\beta^2\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)\Bigg].\end{aligned} \tag{92}$$

The relation

$$\frac{\partial}{\partial\hat{x}_j}\frac{\partial^2}{\partial\hat{x}_{\langle i}\hat{x}_{j\rangle}}(\cdot) = \frac{2}{3}\frac{\partial^2}{\partial\hat{x}_i\partial\hat{x}_j}\frac{\partial}{\partial\hat{x}_j}(\cdot),$$

is also used for replacing the gradients of number densities and temperature in Eqs. (88). Furthermore, the right-hand side of Eq. $(88)_1$ is simplified by using an expression obtained by taking the deviatoric gradient of Eq. $(87)_2$. After all replacements and some algebra, we finally get

$$\begin{aligned}\frac{\partial}{\partial\hat{t}}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right) \approx &-\varepsilon\,\Omega\frac{\mathfrak{a}_5}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\frac{\partial}{\partial\hat{x}_k}\frac{\partial\hat{\sigma}_{k\langle i}}{\partial\hat{x}_{j\rangle}}\\ &- \frac{\mathfrak{a}_6}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\frac{\partial}{\partial\hat{x}_{j\rangle}}\left(\hat{h}_{\langle i} + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right),\end{aligned} \tag{93}$$

$$\begin{aligned}\frac{\partial}{\partial\hat{t}}\left(\hat{h}_i + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}\right) \approx &\,\varepsilon\,\Omega\frac{\mathfrak{a}_2}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}} + \varepsilon\,\Omega\frac{\mathfrak{a}_3}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}\\ &+ \frac{\mathfrak{a}_4}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\frac{\partial}{\partial\hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right),\end{aligned} \tag{94}$$

where the coefficient ς_{32} is also given in appendix C. Therefore, Eqs. (85) on using Eqs. (68),

(73), (76), (93) and (94) yield

$$\frac{\partial \Delta\hat{\sigma}_{ij}^{(2)}}{\partial \hat{t}} \approx \frac{\Omega}{\varpi_4}\left[2\frac{\varsigma_2\varsigma_{11}}{\varkappa_\alpha} + \mathfrak{a}_5\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\}\right]\varepsilon\frac{\partial}{\partial\hat{x}_k}\frac{\partial\hat{\sigma}_{k\langle i}}{\partial\hat{x}_{j\rangle}}$$
$$+ 2\delta_1\frac{\varkappa}{\varkappa_\alpha\varkappa_\beta^2}\frac{\varsigma_{11}}{\varpi_4}\frac{\partial}{\partial\hat{x}_{j\rangle}}\left[\hat{u}_{\langle i}^{(\alpha)} + \mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_{\langle i}} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_{\langle i}} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right)\right]$$
$$+ \frac{\mathfrak{a}_6}{\varpi_4}\left[\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right]\frac{\partial}{\partial\hat{x}_{j\rangle}}\left(\hat{h}_{\langle i} + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right), \tag{95}$$

$$\frac{\partial \Delta\hat{h}_i^{(2)}}{\partial \hat{t}} \approx \frac{\Omega}{\varpi_8}\left[\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right]\left(\mathfrak{a}_2\varepsilon\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}} + \mathfrak{a}_3\varepsilon\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}\right)$$
$$+ \frac{\mathfrak{a}_4}{\varpi_8}\left[\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right]\frac{\partial}{\partial\hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right). \tag{96}$$

On using Eqs. (71), (72), (95) and (96), Eqs. (83) and (84) provide the third order accurate expressions for $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$:

$$\Delta\hat{\sigma}_{ij} \approx -\frac{\varpi_3}{\varpi_4}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right) - \frac{\varepsilon\,\Omega}{\varpi_4}\left(\frac{4}{5}\varsigma_9\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}} + 2\varsigma_{11}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}\right) - \frac{\varepsilon\,\Omega}{\varpi_4}\varsigma_7\frac{\partial\hat{m}_{ijk}}{\partial\hat{x}_k}$$
$$- \varepsilon^2\frac{\Omega^2}{\varpi_4^2}\left[2\frac{\varsigma_2\varsigma_{11}}{\varkappa_\alpha} + \mathfrak{a}_5\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\} - \frac{4}{5}\frac{\varpi_4}{\varpi_8}\varsigma_{10}\varsigma_{15}\right]\frac{\partial}{\partial\hat{x}_k}\frac{\partial\hat{\sigma}_{k\langle i}}{\partial\hat{x}_{j\rangle}}$$
$$- 2\varepsilon\,\Omega\delta_1\frac{\varkappa}{\varkappa_\alpha\varkappa_\beta^2}\frac{\varsigma_{11}}{\varpi_4^2}\frac{\partial}{\partial\hat{x}_{j\rangle}}\left[\hat{u}_{\langle i}^{(\alpha)} + \mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_{\langle i}} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_{\langle i}} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right)\right]$$
$$- \frac{\varepsilon\,\Omega}{\varpi_4}\left[\frac{\mathfrak{a}_6}{\varpi_4}\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\} - \frac{4}{5}\frac{\varpi_7}{\varpi_8}\varsigma_{10}\right]\frac{\partial}{\partial\hat{x}_{j\rangle}}\left(\hat{h}_{\langle i} + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right), \tag{97}$$

$$\Delta\hat{h}_i \approx -\frac{\varpi_7}{\varpi_8}\left(\hat{h}_i + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}\right) - \frac{\varepsilon\,\Omega}{\varpi_8}\varsigma_{15}\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j} - \frac{\varepsilon\,\Omega}{\varpi_8}\left(\frac{1}{2}\varsigma_{17}\frac{\partial\hat{R}_{ij}}{\partial\hat{x}_j} + \frac{1}{6}\varsigma_{19}\frac{\partial\hat{\Delta}}{\partial\hat{x}_i} + \frac{5}{2}\varsigma_{21}\frac{\partial\Delta\hat{T}}{\partial\hat{x}_i}\right)$$
$$- \frac{\varepsilon^2\Omega^2}{\varpi_8}\left[\frac{\mathfrak{a}_2}{\varpi_8}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} - \frac{4}{5}\frac{\varsigma_9\varsigma_{16}}{\varpi_4}\right]\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}$$
$$- \frac{\varepsilon^2\Omega^2}{\varpi_8}\left[\frac{\mathfrak{a}_3}{\varpi_8}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} - 2\frac{\varsigma_{11}\varsigma_{16}}{\varpi_4}\right]\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}$$
$$- \frac{\varepsilon\,\Omega}{\varpi_8}\left[\frac{\mathfrak{a}_4}{\varpi_8}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} - \varsigma_{16}\frac{\varpi_3}{\varpi_4}\right]\frac{\partial}{\partial\hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right). \tag{98}$$

Now, inserting the second order accurate value of $\Delta\hat{\sigma}_{ij}$ from Eq. (71) into Eq. (77), we obtain

ing third order accurate balance equation for the diffusion velocity of the α-constituent

$$\varkappa_\alpha \frac{\partial \hat{u}_i^{(\alpha)}}{\partial \hat{t}} + \varsigma_2 \frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} - \mathfrak{a}_7 \frac{\partial}{\partial \hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta \frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right) - \mathfrak{a}_8 \varepsilon \frac{\partial}{\partial \hat{x}_j}\frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}} - \mathfrak{a}_9 \varepsilon \frac{\partial}{\partial \hat{x}_j}\frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + \mathrm{x}_\beta^\circ \frac{\partial \Delta\hat{T}}{\partial \hat{x}_i}$$
$$= -\delta_1 \frac{1}{\varepsilon\,\Omega}\frac{\varkappa}{\varkappa_\beta^2}\left[\hat{u}_i^{(\alpha)} + \mathrm{x}_\beta^\circ \frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2 \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} - \varkappa_\alpha^2 \frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_i}\right)\right], \quad (99)$$

The coefficients $\mathfrak{a}_7$, $\mathfrak{a}_8$ and $\mathfrak{a}_9$ are given in appendix C. Inserting the second order accurate value of $\Delta\hat{\sigma}_{ij}$ from Eq. (71) and the third order accurate values of $\Delta\hat{\sigma}_{ij}$ and $\Delta\hat{h}_i$ from Eqs. (97) and (98), respectively, into Eqs. (78) and (79), we obtain the third order accurate balance equations for the total stress and the total reduced heat flux:

$$\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{t}} + \left(1 - \frac{\varpi_2}{\varpi_4}\varsigma_7\right)\frac{\partial \hat{m}_{ijk}}{\partial \hat{x}_k} + \mathfrak{a}_2 \frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}} + \mathfrak{a}_3 \frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}}$$
$$- \varepsilon\,\Omega \frac{\varpi_2}{\varpi_4^2}\left[2\frac{\varsigma_2\varsigma_{11}}{\varkappa_\alpha} + \mathfrak{a}_5\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\} - \frac{4}{5}\frac{\varpi_4}{\varpi_8}\varsigma_{10}\varsigma_{15}\right]\frac{\partial}{\partial \hat{x}_k}\frac{\partial \hat{\sigma}_{k\langle i}}{\partial \hat{x}_{j\rangle}}$$
$$- 2\delta_1 \frac{\varkappa}{\varkappa_\alpha \varkappa_\beta^2}\frac{\varsigma_{11}\varpi_2}{\varpi_4^2}\frac{\partial}{\partial \hat{x}_{j\rangle}}\left[\hat{u}_{\langle i}^{(\alpha)} + \mathrm{x}_\beta^\circ \frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2 \frac{\partial \hat{n}_\alpha}{\partial \hat{x}_{\langle i}} - \varkappa_\alpha^2 \frac{\partial \hat{n}_\beta}{\partial \hat{x}_{\langle i}} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_{\langle i}}\right)\right]$$
$$- \frac{\varpi_2}{\varpi_4}\left[\frac{\mathfrak{a}_6}{\varpi_4}\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\} - \frac{4}{5}\frac{\varpi_7}{\varpi_8}\varsigma_{10}\right]\frac{\partial}{\partial \hat{x}_{j\rangle}}\left(\hat{h}_{\langle i} + \varepsilon\kappa\frac{\partial \hat{T}}{\partial \hat{x}_{\langle i}}\right)$$
$$= -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_4\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right), \quad (100)$$

$$\frac{\partial \hat{h}_i}{\partial \hat{t}} + \mathfrak{a}_5\frac{\partial \hat{\sigma}_{ij}}{\partial \hat{x}_j} + \frac{1}{2}\left(1 - \frac{\varpi_6}{\varpi_8}\varsigma_{17}\right)\frac{\partial \hat{R}_{ij}}{\partial \hat{x}_j} + \frac{1}{6}\left(1 - \frac{\varpi_6}{\varpi_8}\varsigma_{19}\right)\frac{\partial \hat{\Delta}}{\partial \hat{x}_i} + \frac{5}{2}\left(\varsigma_{14} - \frac{\varpi_6}{\varpi_8}\varsigma_{21}\right)\frac{\partial \Delta\hat{T}}{\partial \hat{x}_i}$$
$$- \varepsilon\,\Omega\left[\mathfrak{a}_2\frac{\varpi_6}{\varpi_8^2}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} + \frac{4}{5}\frac{\varsigma_9}{\varpi_4}\left(\varsigma_{13} - \frac{\varpi_6}{\varpi_8}\varsigma_{16}\right)\right]\frac{\partial}{\partial \hat{x}_j}\frac{\partial \hat{h}_{\langle i}}{\partial \hat{x}_{j\rangle}}$$
$$- \varepsilon\,\Omega\left[\mathfrak{a}_3\frac{\varpi_6}{\varpi_8^2}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} + 2\frac{\varsigma_{11}}{\varpi_4}\left(\varsigma_{13} - \frac{\varpi_6}{\varpi_8}\varsigma_{16}\right)\right]\frac{\partial}{\partial \hat{x}_j}\frac{\partial \hat{u}_{\langle i}^{(\alpha)}}{\partial \hat{x}_{j\rangle}}$$
$$- \left[\mathfrak{a}_4\frac{\varpi_6}{\varpi_8^2}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} + \frac{\varpi_3}{\varpi_4}\left(\varsigma_{13} - \frac{\varpi_6}{\varpi_8}\varsigma_{16}\right)\right]\frac{\partial}{\partial \hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}\right)$$
$$= -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_6\left(\hat{h}_i + \varepsilon\kappa\frac{\partial \hat{T}}{\partial \hat{x}_i}\right). \quad (101)$$

3. *Regularized 17-moment equations*

The system of regularized 17-moment (R17) equations for binary gas mixtures consists of the conservation laws (62) and the governing equations for $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$, $\hat{h}_i$ (Eqs. (99)–(101))—a total of 17 equations in 3D—and the system is closed with the second order accurate contributions of $\Delta\hat{T}$, $\hat{m}_{ijk}$, $\hat{R}_{ij}$ and $\hat{\Delta}$, given by Eqs. (82). We write the system of regularized

17-moment equations in the closed form below—using Eqs. (82) for the unknowns and the relation $\frac{\partial}{\partial\hat{x}_k}\frac{\partial\hat{\sigma}_{\langle ij}}{\partial\hat{x}_{k\rangle}} = \frac{2}{5}\frac{\partial}{\partial\hat{x}_k}\frac{\partial\hat{\sigma}_{k\langle i}}{\partial\hat{x}_{j\rangle}} + \frac{1}{3}\frac{\partial^2\hat{\sigma}_{ij}}{\partial\hat{x}_k^2}$. It reads

$$\left.\begin{aligned}
\varkappa_\alpha\left(\frac{\partial\hat{n}_\alpha}{\partial\hat{t}} + \frac{\partial\hat{v}_i}{\partial\hat{x}_i}\right) + \frac{\partial\hat{u}_i^{(\alpha)}}{\partial\hat{x}_i} &= 0,\\
\frac{\partial}{\partial\hat{t}}\left(\mu_\alpha \mathrm{x}_\alpha^\circ\hat{n}_\alpha + \mu_\beta\mathrm{x}_\beta^\circ\hat{n}_\beta\right) + \left(\mu_\alpha\mathrm{x}_\alpha^\circ + \mu_\beta\mathrm{x}_\beta^\circ\right)\frac{\partial\hat{v}_i}{\partial\hat{x}_i} &= 0,\\
\varkappa\frac{\partial\hat{v}_i}{\partial\hat{t}} + \frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j} + \mathrm{x}_\alpha^\circ\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_i} + \mathrm{x}_\beta^\circ\frac{\partial\hat{n}_\beta}{\partial\hat{x}_i} + \frac{\partial\hat{T}}{\partial\hat{x}_i} &= 0,\\
\frac{3}{2}\frac{\partial\hat{T}}{\partial\hat{t}} + \frac{\partial\hat{v}_i}{\partial\hat{x}_i} + \frac{\partial\hat{h}_i}{\partial\hat{x}_i} + \varsigma_1\frac{\partial\hat{u}_i^{(\alpha)}}{\partial\hat{x}_i} &= 0,
\end{aligned}\right\} \tag{102}$$

$$\underline{\varkappa_\alpha\frac{\partial\hat{u}_i^{(\alpha)}}{\partial\hat{t}} + \mathfrak{a}_0\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j}} - \underline{\mathfrak{a}_7\frac{\partial}{\partial\hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)} - \underline{\mathfrak{a}_8\varepsilon\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}} - \underline{\mathfrak{a}_9\varepsilon\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}} - \underline{\mathfrak{a}_{10}\varepsilon\frac{\partial}{\partial\hat{x}_i}\frac{\partial\hat{u}_j^{(\alpha)}}{\partial\hat{x}_j}}$$
$$\underline{\underline{- \mathfrak{a}_{11}\varepsilon\frac{\partial}{\partial\hat{x}_i}\frac{\partial\hat{h}_j}{\partial\hat{x}_j}}} = -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_1\left[\hat{u}_i^{(\alpha)} + \mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_i} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_i} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_i}\right)\right], \tag{103}$$

$$\underline{\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{t}} + \mathfrak{a}_2\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}} + \mathfrak{a}_3\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}} - \underline{\mathfrak{a}_{12}\frac{\partial}{\partial\hat{x}_{j\rangle}}\left(\hat{h}_{\langle i} + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right)}$$
$$\underline{- \mathfrak{a}_{13}\frac{\partial}{\partial\hat{x}_{j\rangle}}\left[\hat{u}_{\langle i}^{(\alpha)} + \mathrm{x}_\beta^\circ\frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_{\langle i}} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_{\langle i}} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_{\langle i}}\right)\right]}$$
$$\underline{\underline{- \mathfrak{a}_{14}\varepsilon\frac{\partial}{\partial\hat{x}_k}\frac{\partial\hat{\sigma}_{k\langle i}}{\partial\hat{x}_{j\rangle}} - \mathfrak{a}_{15}\varepsilon\frac{\partial^2\hat{\sigma}_{ij}}{\partial\hat{x}_k^2}}} = -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_4\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right), \tag{104}$$

$$\underline{\frac{\partial\hat{h}_i}{\partial\hat{t}} + \mathfrak{a}_5\frac{\partial\hat{\sigma}_{ij}}{\partial\hat{x}_j}} - \underline{\mathfrak{a}_{16}\frac{\partial}{\partial\hat{x}_j}\left(\hat{\sigma}_{ij} + 2\varepsilon\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right)} - \underline{\mathfrak{a}_{17}\varepsilon\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{h}_{\langle i}}{\partial\hat{x}_{j\rangle}}} - \underline{\mathfrak{a}_{18}\varepsilon\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{u}_{\langle i}^{(\alpha)}}{\partial\hat{x}_{j\rangle}}}$$
$$\underline{\underline{- \mathfrak{a}_{19}\varepsilon\frac{\partial}{\partial\hat{x}_i}\frac{\partial\hat{h}_j}{\partial\hat{x}_j} - \mathfrak{a}_{20}\varepsilon\frac{\partial}{\partial\hat{x}_i}\frac{\partial\hat{u}_j^{(\alpha)}}{\partial\hat{x}_j}}} = -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_6\left(\hat{h}_i + \varepsilon\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}\right). \tag{105}$$

The coefficients $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{20}$ in the R17 equations are also given in appendix C. The underlining in Eqs. (103)–(105) is used for distinguishing terms of different order in ε, the single underline denotes the terms of order $\mathcal{O}(\varepsilon)$ and the double underlines denote the terms of order $\mathcal{O}(\varepsilon^2)$. For zeroth order accuracy, obviously, the conservation laws (102) are closed by setting $\hat{u}_i^{(\alpha)}$, $\hat{\sigma}_{ij}$ and $\hat{h}_i$ to zero, which leads to the Euler equations for binary gas mixtures (61). For first order accuracy in the Knudsen number, it suffices to consider the non-underlined terms in Eqs. (103)–(105), i.e., the terms on the left-hand sides of these equations are set to zero, which leads to the laws of Fick, Navier–Stokes, and Fourier (63).

For second order accuracy, one also needs to consider the single-underlined terms along with right-hand sides in Eqs. (103)–(105), which indeed leads to the second order accurate equations (73)–(76). For third order accuracy, the double underlined terms should also be considered, and then the third order accurate equations are the full R17 equations. Thus, the zeroth, first and second order accurate equations for the binary gas mixtures are inherently contained in the third order accurate R17 equations for binary gas mixtures. Notice that the R17 equations (Eqs. (102)–(105)) do contain the cross-coupling through double underlined terms, for instance, the temperature gradient in the governing equation for $\hat{u}_i^{(\alpha)}$ appears through the reduced heat flux terms on the left-hand side of Eq. (103), and—similarly—the pressure or number density gradients in the governing equation for $\hat{h}_i$ appear through the diffusion velocity terms on the left-hand side of Eq. (105). Thus, in contrast to first and second order accurate equations, the R17 equations can be expected to explain the cross-effects of thermal diffusion and diffusion-thermal in the binary gas mixtures of Maxwell molecules.

Interestingly, in the limiting case when the binary mixture reduces to just a single gas, the R17 equations for binary gas mixtures (Eqs. (102)–(105)) reduce to the well-known R13 equations (in linear form) of Struchtrup and Torrilhon[24] and of Struchtrup[25] for Maxwell molecules. The limiting case arises when either the mole fraction of any component in the mixture is zero (i.e., $\mathrm{x}_\alpha^\circ = 0$ or $\mathrm{x}_\beta^\circ = 0$) or when one component is replaced with the other (i.e., $\beta \to \alpha$ or $\alpha \to \beta$). In all the possible four cases, it suffices to consider the mass balance equation for the mixture $(102)_2$ and, therefore, one can ignore Eq. $(102)_1$. Moreover, the coefficient ς_1 in Eq. $(102)_4$ vanishes that means we do not need the balance equation for the diffusion velocity (103) in the system any more, this agrees with the fact that there should not be any term/equation of the diffusion velocity in the single gas case because there is no diffusion in single gases. Furthermore, owing to same reason, the reduced heat flux $\hat{h}_i$ in the mixture changes to the usual heat flux $\hat{q}_i$ in single gas case. In all these limiting cases, all the coefficients except $\mathfrak{a}_{18}$ and $\mathfrak{a}_{20}$ in Eqs. (104) and (105) immediately reduce to the coefficients in R13 equations for Maxwell molecules, i.e., they reduce to

$$\mathfrak{a}_2 = \mathfrak{a}_{14} = \frac{4}{5}, \quad \mathfrak{a}_3 = \mathfrak{a}_{12} = \mathfrak{a}_{13} = \mathfrak{a}_{16} = 0, \quad \mathfrak{a}_4 = \mathfrak{a}_5 = 1, \quad \mathfrak{a}_6 = \mathfrak{a}_{15} = \frac{2}{3}, \quad \mathfrak{a}_{17} = \frac{12}{5}, \quad \mathfrak{a}_{19} = 2.$$

For $\mathrm{x}_\alpha^\circ = 0$ or $\beta \to \alpha$ or $\alpha \to \beta$ cases, one immediately gets $\mathfrak{a}_{18} = \mathfrak{a}_{20} = 0$. Although, for the case of $\mathrm{x}_\beta^\circ = 0$, both $\mathfrak{a}_{18}$ and $\mathfrak{a}_{20}$ themselves are non-zero, but together with $\hat{u}_i^{(\alpha)}$ they let the whole terms vanish, i.e., $\mathfrak{a}_{18}\hat{u}_i^{(\alpha)} = \mathfrak{a}_{20}\hat{u}_i^{(\alpha)} = 0$, since the diffusion velocities for both the components in the mixture vanish.

VIII. LINEAR STABILITY OF THE EQUATIONS

In order to scrutinize the linear stability of the above sets of equations, we consider them in one dimension and assume plane wave solutions of the form

$$\hat{\boldsymbol{U}} = \hat{\boldsymbol{U}}_0 \exp\{i(\hat{x} - \hat{\omega}\hat{t})\}. \tag{106}$$

In Eq. (106), $\hat{\boldsymbol{U}}$ is the vector containing all field variables in a set of equations; $\hat{\boldsymbol{U}}_0$ contains the complex amplitudes of the corresponding field variables; i is the imaginary unit; $\hat{\omega} = \omega/(k\,v_\circ)$ is the dimensionless frequency of the wave with k being the wavenumber, ω being the complex frequency of the wave, and the velocity scale $v_\circ$ is taken as

$v_\circ = \sqrt{k\,T_\circ/(m_\alpha \mathrm{x}^\circ_\alpha + m_\beta \mathrm{x}^\circ_\beta)}$. The length scale L is taken as the inverse of wavenumber, i.e., $L = 1/k$ so that the Knudsen number is $\varepsilon = \ell\, k$. Owing to this length scale, the Knudsen number now enacts as a dimensionless wavenumber. Insertion of plane wave solution (106) into each system of moment equations yields algebraic equation of the form

$$\mathcal{A}(\hat{\omega}, \varepsilon, \mu_\alpha, \mathrm{x}^\circ_\alpha, \Omega_\alpha, \Omega_\beta)\, \hat{\boldsymbol{U}} = \boldsymbol{0}. \tag{107}$$

For non-trivial solutions of Eq. (107), the determinant of matrix $\mathcal{A}(\hat{\omega}, \varepsilon, \mu_\alpha, \mathrm{x}^\circ_\alpha, \Omega_\alpha, \Omega_\beta)$ must vanish. This condition $\det\{\mathcal{A}(\hat{\omega}, \varepsilon, \mu_\alpha, \mathrm{x}^\circ_\alpha, \Omega_\alpha, \Omega_\beta)\} = 0$ gives the dispersion relation—a relation between ω and κ ($\hat{\omega}$ and ε here). For temporal stability analysis, it is customary to assume a perturbation with real wavenumber k and solve the dispersion relation for complex modes $\omega_j(k) = \mathrm{Re}(\omega_j) + i\ \mathrm{Im}(\omega_j)$, where $j = 1, 2, \ldots, N$ with N being the number of equations in the system considered. Here, we assume $\hat{\omega}_j(\varepsilon) = \mathrm{Re}(\hat{\omega}_j) + i\ \mathrm{Im}(\hat{\omega}_j)$. The growth rate of the amplitude of perturbation is determined by the signs of the imaginary part of $\hat{\omega}_j$ for all j. From Eq. (106), it is clear that for linearly stable solutions, the imaginary part of $\hat{\omega}_j$ must be non-positive for all j.

It is trivial to check that for the zeroth order accurate equations, i.e., for the Euler equations (61), $\mathrm{Im}(\hat{\omega}) = 0$ as long as the wavenumber is real. Thus, the Euler equations (61) are always linearly stable for all gas mixtures. However, for the other sets of equations, owing to the large number of parameters, it is not easy to check the stability for *all* gas mixtures. Nevertheless, we have analyzed the linear stability by considering several different permissible values of the parameters; in particular, we have analyzed the linear stability for three binary mixtures of noble gases: neon–argon (Ne–Ar), helium–argon (He–Ar), helium–xenon (He–Xe). The molecular masses and diameters (for hard spheres) of these gases are listed in Table I. The diameters of these gases (for hard spheres) are calculated using the exact expression of viscosity for a single gas given in Ref. 55 and the experimental data on the viscosities of single gases at temperature 300 K given in Ref. 56. It can be noted that the computation of parameters Ω_α and Ω_β for the Maxwell interaction potential is not so straightforward, since their computation requires explicit viscosity formulas for a single gas as well as for a binary gas mixture and viscosity data from experiments. Although, the viscosity formulas for single gases and binary gas mixtures can be obtained by performing Chapman–Enskog expansion either on the respective Boltzmann equation(s) or on the respective Grad's moment equations, only limited viscosity data from experiments is available in the literature (only for mole fractions 0.25, 0.5 and 0.75 in Ref. 56). Therefore, we compute Ω_α and Ω_β through the values of Omega integrals for hard spheres, i.e., through the relation $\Omega^{(2,2)}_{ij} = (d_i + d_j)^2/4$, where $i, j \in \{\alpha, \beta\}$ and d_i is the molecular diameter gas i so that they can be used for any value of mole fraction, not necessarily 0.25, 0.5 and 0.75. Indeed, Gupta[50] has also computed Ω_α and Ω_β for the Maxwell interaction potential and for mole fractions 0.25, 0.5, 0.75, and found that the values of Ω_α and Ω_β for the Maxwell interaction potential are not very different from those for the hard-sphere interaction potential (or for hard spheres), see Table 5.1 of Ref. 50.

After scrutinizing the linear stability by considering several different permissible values of the parameters, we have found *empirically* that the first order accurate equations (Eqs. (62) and (63))—i.e., Fick, Navier–Stokes, and Fourier equations—as well as the second order accurate equations (Eqs. (73)–(76)) are linearly stable for all binary gas mixtures. As an example, we plot the imaginary part of dimensionless frequency, $\mathrm{Im}(\hat{\omega})$, over the Knudsen number, ε, for He–Xe mixture with $\mathrm{x}^\circ_{\mathrm{He}} = 0.75$ in Figure 1, in which Figure 1(a) illustrates the

TABLE I. Mass and diameters of some noble gases

Gas	Mass (in atomic units)	Diameter (in nanometres)
He	4.0026	0.2166
Ne	20.1791	0.2564
Ar	39.948	0.3606
Xe	131.293	0.4821

(a) Modes from first order accurate equations (Eqs. (62) and (63))

(b) Modes from second order accurate equations (Eqs. (73)–(76))

FIG. 1. Dispersion modes in He–Xe mixture with $x^{\circ}_{He} = 0.75$ obtained with (a) first order accurate (Fick, Navier–Stokes, and Fourier) equations (Eqs. (62) and (63)), and (b) second order accurate equations (Eqs. (73)–(76)).

dispersion modes obtained with the first order accurate (Fick, Navier–Stokes, and Fourier) equations (Eqs. (62) and (63)) while Figure 1(b) depicts the dispersion modes obtained with the second order accurate equations (Eqs. (73)–(76)). Figure 1(a) delineates four modes where two modes coincide with each other for small Knudsen numbers (for $\varepsilon \lesssim 0.323$), while they split into two distinct modes for large Knudsen numbers. Similarly, Figure 1(b) displays seven modes where three modes have quite small damping for very small Knudsen numbers and two modes coincide. For large Knudsen numbers some of the modes coincide with each other. It can be seen from Figure 1 that the imaginary part of the dimensionless frequency remains always non-positive resulting into stability for both first and second order accurate equations.

The R17 equations (Eqs. (102)–(105)), on the other hand, turn out to be stable for the mixtures with small or moderate mass differences but, unfortunately, unstable for the mixtures with large mass differences. We have analyzed the linear stability for three mixtures: Ne–Ar, He–Ar, He–Xe and found that the R17 equations are linearly stable for Ne–Ar and He–Ar mixtures with any mole fraction of the components while they are unstable for He–Xe mixture. As an example, Figure 2 illustrates the dispersion modes from the R17 equations for the three mixtures (a) Ne–Ar, (b) He–Ar and (c) He–Xe with mole fraction of each gas in each mixture being 0.5. Each sub-figure of Figure 2 displays seven modes, where some of

ing modes coincide with each other. It is also clear from Figure 2 that the R17 equations are not stable for He–Xe mixture since it is the mixture with large mass difference but they are stable for the other two mixtures. In order to have more insight into region of instability, we plot the zero contours of maximum value of $\mathrm{Im}(\hat{\omega})$ in Figure 3 which illustrates them in $(\mathrm{x}^\circ_\alpha, \mu_\alpha)$ plane for fixed values of Ω_α and Ω_β: (a) $\Omega_\alpha = 1$, $\Omega_\beta = 1$ and (b) $\Omega_\alpha = 0.5$, $\Omega_\beta = 1.5$. The Knudsen number (ε) varies from 0.001 to 5000. The white color represents the regions in which the R17 equations are stable while the gray color portrays the regions in which they are unstable. Of course, in the limiting cases of $\mu_\alpha \approx 0$ and $\mu_\alpha \approx 1$, the R17 equations reduce to the linearized R13 equations for a single gas and become linearly stable.

(a) Ne–Ar (b) He–Ar (c) He–Xe

FIG. 2. Dispersion modes in different binary gas mixtures computed with R17 equations (Eqs. (102)–(105)): (a) Ne–Ar ($\mathrm{x}^\circ_{\mathrm{Ne}} = 0.5$), (b) He–Ar ($\mathrm{x}^\circ_{\mathrm{He}} = 0.5$) and (c) He–Xe ($\mathrm{x}^\circ_{\mathrm{He}} = 0.5$).

(a) $\Omega_\alpha = 1$ and $\Omega_\beta = 1$ (b) $\Omega_\alpha = 0.5$ and $\Omega_\beta = 1.5$

FIG. 3. Zero contours of maximum (dimensionless) damping for different mole fractions (x°_α) and different mass ratios (μ_α) with fixed values of Ω_α and Ω_β: (a) $\Omega_\alpha = 1$, $\Omega_\beta = 1$ and (b) $\Omega_\alpha = 0.5$, $\Omega_\beta = 1.5$. The Knudsen number (ε) varies from 0.001 to 5000. The white color represents the regions in which the R17 equations are stable while the gray color portrays the regions in which they are unstable.

It can be stated from Figure 3 that for plausible values of Ω_α and Ω_β, the R17 equations seem to be stable for the mixtures with mass ratios $0.1 \lesssim \mu_\alpha \lesssim 0.9$ for any mole fractions.

IX. ONE-DIMENSIONAL APPLICATION OF THE R17 EQUATIONS

In order to solve boundary value problems with the moment equations derived in Section VII, these equations must be supplemented with proper boundary conditions. However, as mentioned before, the derivation of proper boundary conditions for the R17 equations equations is beyond the scope of the present paper and will be considered elsewhere in the future. Therefore, in this section, we consider a very simple one-dimensional flow problem of binary gas mixture in steady state, which we can study with the R17 equations analytically without boundary conditions. The R17 equations (considering the flow in x-direction) for

this problem simplifies to

$$\left.\begin{aligned} \frac{\mathrm{d}\hat{u}_x^{(\alpha)}}{\mathrm{d}\hat{x}} &= 0, \\ \frac{\mathrm{d}\hat{v}_x}{\mathrm{d}\hat{x}} &= 0, \\ \frac{\mathrm{d}\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}} + \mathrm{x}_\alpha^\circ \frac{\mathrm{d}\hat{n}_\alpha}{\mathrm{d}\hat{x}} + \mathrm{x}_\beta^\circ \frac{\mathrm{d}\hat{n}_\beta}{\mathrm{d}\hat{x}} + \frac{\mathrm{d}\hat{T}}{\mathrm{d}\hat{x}} &= 0, \\ \frac{\mathrm{d}\hat{h}_x}{\mathrm{d}\hat{x}} &= 0, \end{aligned}\right\} \tag{108}$$

$$\underline{\mathfrak{a}_0 \frac{\mathrm{d}\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}}} - \underline{\underline{\mathfrak{a}_7 \frac{\mathrm{d}\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}}}} = -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_1 \left[\hat{u}_x^{(\alpha)} + \mathrm{x}_\beta^\circ \frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2 \frac{\mathrm{d}\hat{n}_\alpha}{\mathrm{d}\hat{x}} - \varkappa_\alpha^2 \frac{\mathrm{d}\hat{n}_\beta}{\mathrm{d}\hat{x}} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\mathrm{d}\hat{T}}{\mathrm{d}\hat{x}} \right) \right], \tag{109}$$

$$\underline{\underline{-\frac{2}{3}\mathfrak{a}_{12}\frac{\mathrm{d}}{\mathrm{d}\hat{x}}\left(\hat{h}_x + \varepsilon\kappa \frac{\mathrm{d}\hat{T}}{\mathrm{d}\hat{x}} \right) - \frac{2}{3}\mathfrak{a}_{13}\frac{\mathrm{d}}{\mathrm{d}\hat{x}}\left[\hat{u}_x^{(\alpha)} + \mathrm{x}_\beta^\circ \frac{\varepsilon\,\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2 \frac{\mathrm{d}\hat{n}_\alpha}{\mathrm{d}\hat{x}} - \varkappa_\alpha^2 \frac{\mathrm{d}\hat{n}_\beta}{\mathrm{d}\hat{x}} - (\varkappa_\alpha^2 - \varkappa_\beta^2)\frac{\mathrm{d}\hat{T}}{\mathrm{d}\hat{x}} \right) \right]}}$$

$$\underline{\underline{-\frac{2}{3}\mathfrak{a}_{14}\varepsilon\frac{\mathrm{d}^2\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}^2} - \mathfrak{a}_{15}\varepsilon\frac{\mathrm{d}^2\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}^2}}} = -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_4\hat{\sigma}_{xx}, \tag{110}$$

$$\underline{\mathfrak{a}_5 \frac{\mathrm{d}\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}}} - \underline{\underline{\mathfrak{a}_{16} \frac{\mathrm{d}\hat{\sigma}_{xx}}{\mathrm{d}\hat{x}}}} = -\frac{1}{\varepsilon\,\Omega}\mathfrak{a}_6 \left(\hat{h}_x + \varepsilon\kappa \frac{\mathrm{d}\hat{T}}{\mathrm{d}\hat{x}} \right). \tag{111}$$

Eqs. (108)–(111) form a system of first order ordinary differential equations, whose solution turns out to be

$$\left.\begin{aligned} \hat{u}_x^{(\alpha)}(\hat{x}) = c_1, \quad \hat{v}_x(\hat{x}) = c_2, \quad \hat{h}_x(\hat{x}) = c_3, \quad \hat{\sigma}_{xx}(\hat{x}) &= c_4\,\mathrm{e}^{A\hat{x}} + c_5\,\mathrm{e}^{-A\hat{x}}, \\ \hat{T}(\hat{x}) = c_6 - \frac{c_3}{\varepsilon\kappa}\hat{x} - \frac{(\mathfrak{a}_5 - \mathfrak{a}_{16})}{\mathfrak{a}_6}\frac{\Omega}{\kappa}\hat{\sigma}_{xx}(\hat{x}), & \\ \hat{n}_\alpha(\hat{x}) = c_7 - c_1\frac{1}{\varepsilon\,\Omega}\delta_1\frac{\varkappa}{\varkappa_\beta^2}\hat{x} - \hat{T}(\hat{x}) - \left[\frac{(\mathfrak{a}_0 - \mathfrak{a}_7)}{\mathfrak{a}_1}\delta_1\frac{\varkappa}{\varkappa_\beta^2} + \frac{\varkappa_\alpha^2}{\varkappa} \right]\hat{\sigma}_{xx}(\hat{x}), & \\ \hat{n}_\beta(\hat{x}) = c_8 - \frac{1}{\mathrm{x}_\beta^\circ}\left[\mathrm{x}_\alpha^\circ\,\hat{n}_\alpha(\hat{x}) + \hat{T}(\hat{x}) + \hat{\sigma}_{xx}(\hat{x}) \right], & \end{aligned}\right\} \tag{112}$$

where $c_1, c_2, \ldots, c_8$ are the integration constants—and can be computed through boundary conditions—and

$$A = \frac{1}{\varepsilon}\sqrt{\frac{\mathfrak{a}_4}{\Omega\left[\frac{2}{3}\mathfrak{a}_{14} + \mathfrak{a}_{15} - \frac{2}{3}\Omega\left\{ \frac{\mathfrak{a}_{13}}{\mathfrak{a}_1}(\mathfrak{a}_0 - \mathfrak{a}_7) + \frac{\mathfrak{a}_{12}}{\mathfrak{a}_6}(\mathfrak{a}_5 - \mathfrak{a}_{16}) \right\} \right]}}. \tag{113}$$

Notice that the solution (112) is a combination of bulk solution and Knudsen boundary layers—which are given by the superposition of exponential functions and are well-known rarefaction effect. The bulk solution follows from the first order accurate equations (i.e., when single and double underlined terms in Eqs. (109)–(111) are zero) and reads

$$\left.\begin{aligned} \hat{u}_x^{(\alpha)}(\hat{x}) = c_1, \quad \hat{v}_x(\hat{x}) = c_2, \quad \hat{h}_x(\hat{x}) = c_3, \quad \hat{\sigma}_{xx}(\hat{x}) = 0, \quad \hat{T}(\hat{x}) = c_6 - \frac{c_3}{\varepsilon\kappa}\hat{x}, \\ \hat{n}_\alpha(\hat{x}) = c_7 - c_6 + \left(\frac{c_3}{\varepsilon\kappa} - c_1\frac{1}{\varepsilon\,\Omega}\delta_1\frac{\varkappa}{\varkappa_\beta^2}\right)\hat{x}, \\ \hat{n}_\beta(\hat{x}) = c_8 - \frac{1}{\mathrm{x}_\beta^\circ}\left[\mathrm{x}_\alpha^\circ c_7 + \mathrm{x}_\beta^\circ c_6 - \left(\mathrm{x}_\beta^\circ\frac{c_3}{\varepsilon\kappa} + \mathrm{x}_\alpha^\circ c_1\frac{1}{\varepsilon\,\Omega}\delta_1\frac{\varkappa}{\varkappa_\beta^2}\right)\hat{x}\right]. \end{aligned}\right\} \tag{114}$$

Thus, in the bulk of the problem domain, the number densities of the constituents and the temperature of mixture have linear profiles, the stress vanishes, and the diffusion velocities of the constituents, the mean flow velocity and the (reduced) heat flux are constants. Needless to say, the first order accurate equations cannot capture the well-known Knudsen boundary layers (which are typically superposition of exponential functions). Furthermore, it may be noted that—for this problem—the Knudsen boundary layers in the solution appear only through the stress (see Eqs. (112)), whose governing equation does not contain single underlined terms (see Eqs. (110)). This means that the second order accurate equations also lead to same solution as that obtained with the first order accurate equations for this simple problem. In other words, for this simple problem, even the second order accurate equations cannot capture the Knudsen boundary layers. Nevertheless, the—third order accurate—R17 equations do capture the Knudsen boundary layers (see Eqs. (112)).

X. CONCLUSION AND DISCUSSION

In this paper, the applicability of order of magnitude method has been extended to binary gas mixtures in order to derive various sets of equations for binary mixtures of monatomic-inert-ideal gases interacting with the Maxwell interaction potential up to third accuracy in the Knudsen number. For simplicity, the equations have been derived only in linear regime. To zeroth order accuracy, the method has resulted into the Euler equations; to first order accuracy, it has led to the Fick, Navier–Stokes, and Fourier equations; at second order, it has yielded the 17 moment equations; and at third order it has ensued the R17 equations. The transport coefficients in Fick, Navier–Stokes, and Fourier equations obtained through the order of magnitude method have been compared and matched with those obtained through the classical Chapman–Enskog expansion. It has been established that the temperature difference does not play any role up to second order accurate theories in the Knudsen number, however, it does become important for a third order theory in the Knudsen number. Furthermore, it has also been found that the cross-effects of thermal diffusion and diffusion-thermal in binary gas mixtures of Maxwell molecules cannot be captured by the first and second order accurate equations; nonetheless, the R17 equations can be expected to capture these effects due to cross-coupling terms present in the respective equations. The linear stability of the derived equations has been analyzed and it has been found *empirically* that the Euler equations, Navier–Stokes, and Fourier equations, and second order accurate 17 moment equations are linearly stable for any binary gas mixture whereas the R17 equations

ing unstable for mixtures having extreme differences in molecular masses, although they are linearly stable for other mixtures. Finally, a one-dimensional steady state flow problem of binary gas mixtures of Maxwell molecules has been investigated through the various sets of derived equations, and it has been shown that the first and second order accurate equations cannot describe the Knudsen boundary layers for this simple problem but the R17 equations do so.

Indeed, on investigating closely, we have found that the third order accurate 25 equations (Eqs. (62) and (77)–(82)) at intermediate step themselves are not stable for any of the three mixtures above but the R17 equations have, luckily, turned out to be linearly stable for mixtures with small and moderate mass differences. We have noticed that if we include the equation for temperature difference with these 25 equations at intermediate step, the resulting 26 equations become linearly stable at least for the three mixtures considered above. However, the R17 equations supplemented with the equation for the temperature difference are still unstable for these mixtures. Thus, the linear stability of equations also depends on the choice of moments. We have also scrutinized the case of single temperature theory. In this case, the third order accurate 25 equations (Eqs. (62) and (77)–(82)) are linearly stable at least for the three mixtures considered above. However, the R17 equations are again unstable for He–Xe mixture, although they are linearly stable for Ne–Ar and He–Ar mixtures.

It is worth pointing out that for mixtures with extreme differences in molecular masses, one of the two mass ratios is obviously very small, and in fact, it can well be of the order of Knudsen number or even smaller. Therefore, while applying the order of magnitude method for mixtures with extreme differences in molecular masses, one has to take care of mass ratios since one of them would contribute to some power of the Knudsen number in each term wherever it appears. At this point, it is very complicated to deal with this issue and one would need to rederive the R17 equations for mixtures with extreme mass differences separately by considering this issue.

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Appendix A: Coefficients in linear-dimensionless Grad's moment equations

$$\delta_1 = \frac{2\sqrt{2}}{3} a_1 \sqrt{\mu_\beta},$$
$$\delta_2 = 2\sqrt{2}\, a_1 \mu_\alpha \sqrt{\mu_\beta},$$
$$\delta_3 = \frac{\sqrt{2}}{3}(4a_1\mu_\alpha + 3\mu_\beta)\sqrt{\mu_\beta},$$
$$\delta_4 = \frac{\sqrt{2}}{3}(4a_1 - 3)\mu_\alpha \sqrt{\mu_\beta},$$

$$\delta_5 = \frac{2\sqrt{2}}{3}(3a_1\mu_\alpha^2 + 2\mu_\alpha\mu_\beta + a_1\mu_\beta^2)\sqrt{\mu_\beta},$$
$$\delta_6 = \frac{4\sqrt{2}}{3}(2a_1 - 1)\mu_\alpha\mu_\beta\sqrt{\mu_\alpha},$$
$$\delta_7 = \frac{\sqrt{2}}{3}\Big\{6a_1\mu_\alpha^2 + 9\mu_\alpha\mu_\beta + (5a_3 - 3a_1)\mu_\beta^2\Big\}\sqrt{\mu_\beta},$$
$$\delta_8 = \frac{\sqrt{2}}{3}(5a_3 + 3a_1 - 9)\mu_\alpha\mu_\beta\sqrt{\mu_\alpha},$$
$$\delta_9 = \frac{\sqrt{2}}{3}\Big[3\mu_\beta + 2\mu_\alpha\Big\{4a_1\mu_\alpha^2 + 4\mu_\alpha\mu_\beta + (3a_3 + a_1 - 3)\mu_\beta^2\Big\}\Big]\sqrt{\mu_\beta},$$
$$\delta_{10} = \frac{2\sqrt{2}}{3}(3a_3 + 5a_1 - 7)\mu_\alpha^2\mu_\beta\sqrt{\mu_\beta},$$
$$\delta_{11} = \frac{8\sqrt{2}}{3}\mu_\alpha(a_1\mu_\alpha^2 + \mu_\alpha\mu_\beta + a_1\mu_\beta^2)\sqrt{\mu_\beta},$$
$$\delta_{12} = \frac{8\sqrt{2}}{3}(2a_1 - 1)\mu_\alpha^2\mu_\beta\sqrt{\mu_\beta},$$

$$\gamma_1 = \frac{2\sqrt{2}}{3}a_1\sqrt{\mu_\alpha},$$
$$\gamma_2 = 2\sqrt{2}\,a_1\mu_\beta\sqrt{\mu_\alpha},$$
$$\gamma_3 = \frac{\sqrt{2}}{3}(4a_1\mu_\beta + 3\mu_\alpha)\sqrt{\mu_\alpha},$$
$$\gamma_4 = \frac{\sqrt{2}}{3}(4a_1 - 3)\mu_\beta\sqrt{\mu_\alpha},$$
$$\gamma_5 = \frac{2\sqrt{2}}{3}(3a_1\mu_\beta^2 + 2\mu_\beta\mu_\alpha + a_1\mu_\alpha^2)\sqrt{\mu_\alpha},$$
$$\gamma_6 = \frac{4\sqrt{2}}{3}(2a_1 - 1)\mu_\beta\mu_\alpha\sqrt{\mu_\beta},$$
$$\gamma_7 = \frac{\sqrt{2}}{3}\Big\{6a_1\mu_\beta^2 + 9\mu_\beta\mu_\alpha + (5a_3 - 3a_1)\mu_\alpha^2\Big\}\sqrt{\mu_\alpha},$$
$$\gamma_8 = \frac{\sqrt{2}}{3}(5a_3 + 3a_1 - 9)\mu_\beta\mu_\alpha\sqrt{\mu_\beta},$$
$$\gamma_9 = \frac{\sqrt{2}}{3}\Big[3\mu_\alpha + 2\mu_\beta\Big\{4a_1\mu_\beta^2 + 4\mu_\beta\mu_\alpha + (3a_3 + a_1 - 3)\mu_\alpha^2\Big\}\Big]\sqrt{\mu_\alpha},$$
$$\gamma_{10} = \frac{2\sqrt{2}}{3}(3a_3 + 5a_1 - 7)\mu_\beta^2\mu_\alpha\sqrt{\mu_\alpha},$$
$$\gamma_{11} = \frac{8\sqrt{2}}{3}\mu_\beta(a_1\mu_\beta^2 + \mu_\beta\mu_\alpha + a_1\mu_\alpha^2)\sqrt{\mu_\alpha},$$
$$\gamma_{12} = \frac{8\sqrt{2}}{3}(2a_1 - 1)\mu_\beta^2\mu_\alpha\sqrt{\mu_\alpha},$$

Appendix B: Leading orders of higher moments

Comparing coefficients of ε^0 on both sides of Eqs. (20) and (18)–(30), one obtains:

$$\Delta \hat{T}_{|1} = 0, \tag{B1}$$

$$\left.\begin{aligned} \frac{1}{\varkappa}\left[\varkappa_\beta^2\frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} - \varkappa_\alpha^2\frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} - (\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial \hat{T}}{\partial \hat{x}_i}\right] &= -\frac{\delta_1}{\Omega}\left(\hat{u}_{i|1}^{(\alpha)} - \frac{\varkappa_\alpha}{\varkappa_\beta}\hat{u}_{i|1}^{(\beta)}\right), \\ \frac{1}{\varkappa}\left[\varkappa_\alpha^2\frac{\partial \hat{n}_\beta}{\partial \hat{x}_i} - \varkappa_\beta^2\frac{\partial \hat{n}_\alpha}{\partial \hat{x}_i} - (\varkappa_\beta^2-\varkappa_\alpha^2)\frac{\partial \hat{T}}{\partial \hat{x}_i}\right] &= -\frac{\gamma_1}{\Omega}\left(\hat{u}_{i|1}^{(\beta)} - \frac{\varkappa_\beta}{\varkappa_\alpha}\hat{u}_{i|1}^{(\alpha)}\right), \end{aligned}\right\} \tag{B2}$$

$$\left.\begin{aligned} 2\varkappa_\alpha\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}} &= -\frac{1}{\Omega}\left\{\mathrm{x}_\alpha^\circ\Omega_\alpha\hat{\sigma}_{ij|1}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_3\hat{\sigma}_{ij|1}^{(\alpha)} - \delta_4\hat{\sigma}_{ij|1}^{(\beta)}\right)\right\}, \\ 2\varkappa_\beta\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}} &= -\frac{1}{\Omega}\left\{\mathrm{x}_\beta^\circ\Omega_\beta\hat{\sigma}_{ij|1}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_3\hat{\sigma}_{ij|1}^{(\beta)} - \gamma_4\hat{\sigma}_{ij|1}^{(\alpha)}\right)\right\}, \end{aligned}\right\} \tag{B3}$$

$$\left.\begin{aligned} \frac{5}{2}\frac{\partial \hat{T}}{\partial \hat{x}_i} &= -\frac{1}{\Omega}\left\{\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha\hat{h}_{i|1}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_5\hat{h}_{i|1}^{(\alpha)} - \delta_6\hat{h}_{i|1}^{(\beta)}\right)\right\}, \\ \frac{5}{2}\frac{\partial \hat{T}}{\partial \hat{x}_i} &= -\frac{1}{\Omega}\left\{\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta\hat{h}_{i|1}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_5\hat{h}_{i|1}^{(\beta)} - \gamma_6\hat{h}_{i|1}^{(\alpha)}\right)\right\}, \end{aligned}\right\} \tag{B4}$$

$$\left.\begin{aligned} 0 &= -\frac{1}{\Omega}\left\{\frac{3}{2}\mathrm{x}_\alpha^\circ\Omega_\alpha\hat{m}_{ijk|1}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_7\hat{m}_{ijk|1}^{(\alpha)} - \delta_8\hat{m}_{ijk|1}^{(\beta)}\right)\right\}, \\ 0 &= -\frac{1}{\Omega}\left\{\frac{3}{2}\mathrm{x}_\beta^\circ\Omega_\beta\hat{m}_{ijk|1}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_7\hat{m}_{ijk|1}^{(\beta)} - \gamma_8\hat{m}_{ijk|1}^{(\alpha)}\right)\right\}, \end{aligned}\right\} \tag{B5}$$

$$\left.\begin{aligned} 0 &= -\frac{1}{\Omega}\left\{\frac{7}{6}\mathrm{x}_\alpha^\circ\Omega_\alpha\hat{R}_{ij|1}^{(\alpha)} + \mathrm{x}_\beta^\circ\left(\delta_9\hat{R}_{ij|1}^{(\alpha)} - \delta_{10}\hat{R}_{ij|1}^{(\beta)}\right)\right\}, \\ 0 &= -\frac{1}{\Omega}\left\{\frac{7}{6}\mathrm{x}_\beta^\circ\Omega_\beta\hat{R}_{ij|1}^{(\beta)} + \mathrm{x}_\alpha^\circ\left(\gamma_9\hat{R}_{ij|1}^{(\beta)} - \gamma_{10}\hat{R}_{ij|1}^{(\alpha)}\right)\right\}, \end{aligned}\right\} \tag{B6}$$

$$\left.\begin{aligned} 0 &= -\frac{1}{\Omega}\left\{\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha\hat{\Delta}_{\alpha|1} + \mathrm{x}_\beta^\circ\left(\delta_{11}\hat{\Delta}_{\alpha|1} - \delta_{12}\hat{\Delta}_{\beta|1}\right)\right\}, \\ 0 &= -\frac{1}{\Omega}\left\{\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta\hat{\Delta}_{\beta|1} + \mathrm{x}_\alpha^\circ\left(\gamma_{11}\hat{\Delta}_{\beta|1} - \gamma_{12}\hat{\Delta}_{\alpha|1}\right)\right\}. \end{aligned}\right\} \tag{B7}$$

Eqs. (B1)–(B7) on using the relation $\rho_\alpha \boldsymbol{u}_\alpha + \rho_\beta \boldsymbol{u}_\beta = \boldsymbol{0}$ in dimensionless form imply that $\hat{u}^{(\alpha)}_{i|1}$, $\hat{u}^{(\beta)}_{i|1}$, $\hat{\sigma}^{(\alpha)}_{ij|1}$, $\hat{\sigma}^{(\beta)}_{ij|1}$, $\hat{h}^{(\alpha)}_{i|1}$, $\hat{h}^{(\beta)}_{i|1}$ do not vanish, whereas

$$\Delta\hat{T}_{|1} = \hat{m}^{(\alpha)}_{ijk|1} = \hat{m}^{(\beta)}_{ijk|1} = \hat{R}^{(\alpha)}_{ij|1} = \hat{R}^{(\beta)}_{ij|1} = \hat{\Delta}_{\alpha|1} = \hat{\Delta}_{\beta|1} = 0. \tag{B8}$$

Thus, the leading orders of the diffusion velocities, stresses and heat fluxes of both the constituents are one while the leading orders of temperature difference and other higher moments for both the constituents are at least two.

Comparing the coefficients of ε^1 on both sides of Eqs. (20) and (25)–(30), one obtains

$$\frac{1}{\varkappa_\alpha}\left(\frac{\partial \hat{h}^{(\alpha)}_{i|1}}{\partial \hat{x}_i} + \frac{\partial \hat{u}^{(\alpha)}_{i|1}}{\partial \hat{x}_i}\right) - \frac{1}{\varkappa_\beta}\left(\frac{\partial \hat{h}^{(\beta)}_{i|1}}{\partial \hat{x}_i} + \frac{\partial \hat{u}^{(\beta)}_{i|1}}{\partial \hat{x}_i}\right) = -\frac{1}{\Omega}\frac{\delta_2}{\varkappa_\alpha}\Delta\hat{T}_{|2}, \tag{B9}$$

$$\left.\begin{aligned} 3\frac{\partial \hat{\sigma}^{(\alpha)}_{\langle ij|1}}{\partial \hat{x}_{k\rangle}} &= -\frac{1}{\Omega}\left\{\frac{3}{2}\mathrm{x}^\circ_\alpha\Omega_\alpha\hat{m}^{(\alpha)}_{ijk|2} + \mathrm{x}^\circ_\beta\left(\delta_7\hat{m}^{(\alpha)}_{ijk|2} - \delta_8\hat{m}^{(\beta)}_{ijk|2}\right)\right\}, \\ 3\frac{\partial \hat{\sigma}^{(\beta)}_{\langle ij|1}}{\partial \hat{x}_{k\rangle}} &= -\frac{1}{\Omega}\left\{\frac{3}{2}\mathrm{x}^\circ_\beta\Omega_\beta\hat{m}^{(\beta)}_{ijk|2} + \mathrm{x}^\circ_\alpha\left(\gamma_7\hat{m}^{(\beta)}_{ijk|2} - \gamma_8\hat{m}^{(\alpha)}_{ijk|2}\right)\right\}, \end{aligned}\right\} \tag{B10}$$

$$\left.\begin{aligned} \frac{28}{5}\frac{\partial \hat{h}^{(\alpha)}_{\langle i|1}}{\partial \hat{x}_{j\rangle}} &= -\frac{1}{\Omega}\left\{\frac{7}{6}\mathrm{x}^\circ_\alpha\Omega_\alpha\hat{R}^{(\alpha)}_{ij|2} + \mathrm{x}^\circ_\beta\left(\delta_9\hat{R}^{(\alpha)}_{ij|2} - \delta_{10}\hat{R}^{(\beta)}_{ij|2}\right)\right\}, \\ \frac{28}{5}\frac{\partial \hat{h}^{(\beta)}_{\langle i|1}}{\partial \hat{x}_{j\rangle}} &= -\frac{1}{\Omega}\left\{\frac{7}{6}\mathrm{x}^\circ_\beta\Omega_\beta\hat{R}^{(\beta)}_{ij|2} + \mathrm{x}^\circ_\alpha\left(\gamma_9\hat{R}^{(\beta)}_{ij|2} - \gamma_{10}\hat{R}^{(\alpha)}_{ij|2}\right)\right\}, \end{aligned}\right\} \tag{B11}$$

$$\left.\begin{aligned} 8\frac{\partial \hat{h}^{(\alpha)}_{i|1}}{\partial \hat{x}_i} &= -\frac{1}{\Omega}\left\{\frac{2}{3}\mathrm{x}^\circ_\alpha\Omega_\alpha\hat{\Delta}_{\alpha|2} + \mathrm{x}^\circ_\beta\left(\delta_{11}\hat{\Delta}_{\alpha|2} - \delta_{12}\hat{\Delta}_{\beta|2}\right)\right\}, \\ 8\frac{\partial \hat{h}^{(\beta)}_{i|1}}{\partial \hat{x}_i} &= -\frac{1}{\Omega}\left\{\frac{2}{3}\mathrm{x}^\circ_\beta\Omega_\beta\hat{\Delta}_{\beta|2} + \mathrm{x}^\circ_\alpha\left(\gamma_{11}\hat{\Delta}_{\beta|2} - \gamma_{12}\hat{\Delta}_{\alpha|2}\right)\right\}. \end{aligned}\right\} \tag{B12}$$

From Eqs. (B10)–(B12), it is clear that $\hat{m}^{(\alpha)}_{ijk|2}$, $\hat{m}^{(\beta)}_{ijk|2}$, $\hat{R}^{(\alpha)}_{ij|2}$, $\hat{R}^{(\beta)}_{ij|2}$, $\hat{\Delta}_{\alpha|2}$, and $\hat{\Delta}_{\beta|2}$ do not vanish and therefore the leading orders of these quantities are two. Also, one can verify from Eq. (B9)—by inserting the values of $\hat{u}^{(\alpha)}_{i|1}$, $\hat{u}^{(\beta)}_{i|1}$, $\hat{h}^{(\alpha)}_{i|1}$ and $\hat{h}^{(\beta)}_{i|1}$ from Eqs. (34) and (36)—that $\Delta\hat{T}_{|2}$ is also non-zero. Therefore, the leading order of $\Delta\hat{T}_{|2}$ is two as well.

Appendix C: Coefficients in other equations

The coefficients in Eqs. (40) are as follows.

$$\left.\begin{aligned} c_m^{(\alpha)} &= \frac{3\,\Omega}{(\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1)}\left[\frac{\kappa_2\left(\frac{3}{2}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_7\right)+\kappa_1\mathrm{x}_\beta^\circ\delta_8}{\left(\frac{3}{2}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_7\right)\left(\frac{3}{2}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_7\right)-\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\delta_8\gamma_8}\right],\\ c_m^{(\beta)} &= \frac{3\,\Omega}{(\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1)}\left[\frac{\kappa_1\left(\frac{3}{2}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_7\right)+\kappa_2\mathrm{x}_\alpha^\circ\gamma_8}{\left(\frac{3}{2}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_7\right)\left(\frac{3}{2}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_7\right)-\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\delta_8\gamma_8}\right].\end{aligned}\right\} \tag{C1}$$

$$\left.\begin{aligned} c_R^{(\alpha)} &= \frac{28}{5}\Omega\frac{\varkappa_\alpha\varkappa_\beta}{(\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha)}\left[\frac{\kappa_4\left(\frac{7}{6}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_9\right)+\kappa_3\mathrm{x}_\beta^\circ\delta_{10}}{\left(\frac{7}{6}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_9\right)\left(\frac{7}{6}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_9\right)-\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\delta_{10}\gamma_{10}}\right],\\ c_R^{(\beta)} &= \frac{28}{5}\Omega\frac{\varkappa_\alpha\varkappa_\beta}{(\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha)}\left[\frac{\kappa_3\left(\frac{7}{6}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_9\right)+\kappa_4\mathrm{x}_\alpha^\circ\gamma_{10}}{\left(\frac{7}{6}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_9\right)\left(\frac{7}{6}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_9\right)-\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\delta_{10}\gamma_{10}}\right].\end{aligned}\right\} \tag{C2}$$

$$\left.\begin{aligned} c_\Delta^{(\alpha)} &= 8\Omega\frac{\varkappa_\alpha\varkappa_\beta}{(\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha)}\left[\frac{\kappa_4\left(\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_{11}\right)+\kappa_3\mathrm{x}_\beta^\circ\delta_{12}}{\left(\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_{11}\right)\left(\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_{11}\right)-\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\delta_{12}\gamma_{12}}\right],\\ c_\Delta^{(\beta)} &= 8\Omega\frac{\varkappa_\alpha\varkappa_\beta}{(\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha)}\left[\frac{\kappa_3\left(\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_{11}\right)+\kappa_4\mathrm{x}_\alpha^\circ\gamma_{12}}{\left(\frac{2}{3}\mathrm{x}_\alpha^\circ\Omega_\alpha+\mathrm{x}_\beta^\circ\delta_{11}\right)\left(\frac{2}{3}\mathrm{x}_\beta^\circ\Omega_\beta+\mathrm{x}_\alpha^\circ\gamma_{11}\right)-\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\delta_{12}\gamma_{12}}\right].\end{aligned}\right\} \tag{C3}$$

The coefficients ζ_m, ζ_R and ζ_Δ are as follows.

$$\left.\begin{aligned} \zeta_m &= 3\Omega\frac{\varsigma_{12}}{\varpi_9} = \frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha}c_m^{(\alpha)}+\frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta}c_m^{(\beta)},\\ \zeta_R &= \frac{28}{5}\Omega\frac{\varsigma_{33}}{\varpi_{13}} = \frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha^2}c_R^{(\alpha)}+\frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta^2}c_R^{(\alpha)},\\ \zeta_\Delta &= 8\Omega\frac{\varsigma_{33}}{\varpi_{17}} = \frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha^2}c_\Delta^{(\alpha)}+\frac{\mathrm{x}_\beta^\circ}{\varkappa_\beta^2}c_\Delta^{(\alpha)}.\end{aligned}\right\} \tag{C4}$$

The coefficients ς_i's are as follows.

$$\varsigma_1=\frac{\mathrm{x}_\alpha^\circ}{\varkappa_\alpha}\left(1-\frac{\varkappa_\alpha^2}{\varkappa_\beta^2}\right),\quad \varsigma_2=\frac{\mathrm{x}_\beta^\circ(\kappa_2\varkappa_\beta^2-\kappa_1\varkappa_\alpha^2)}{\varkappa(\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1)},\quad \varsigma_3=\frac{\mathrm{x}_\beta^\circ}{\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1},\quad \varsigma_4=\frac{\kappa_4\varkappa_\beta-\kappa_3\varkappa_\alpha}{\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha},$$

$$\varsigma_5=\frac{1}{\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha},\quad \varsigma_6=\frac{\varkappa}{\mathrm{x}_\beta^\circ\varkappa_\alpha\varkappa_\beta^2},\quad \varsigma_7=\frac{\kappa_1\kappa_6\varkappa_\beta-\kappa_2\kappa_5\varkappa_\alpha}{\mathrm{x}_\alpha^\circ\kappa_6\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_5\varkappa_\alpha},\quad \varsigma_8=\frac{\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1}{\mathrm{x}_\alpha^\circ\kappa_6\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_5\varkappa_\alpha},$$

$$\varsigma_9=\frac{\kappa_1\kappa_4\varkappa_\beta-\kappa_2\kappa_3\varkappa_\alpha}{\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha},\quad \varsigma_{10}=\frac{\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1}{\mathrm{x}_\alpha^\circ\kappa_4\varkappa_\beta+\mathrm{x}_\beta^\circ\kappa_3\varkappa_\alpha},\quad \varsigma_{11}=\frac{\mathrm{x}_\alpha^\circ\kappa_2\varkappa_\alpha^2+\mathrm{x}_\beta^\circ\kappa_1\varkappa_\beta^2}{\mathrm{x}_\beta^\circ\varkappa_\alpha\varkappa_\beta^2},$$

$$\varsigma_{12}=\frac{\mathrm{x}_\alpha^\circ\kappa_2\varkappa_\beta^2+\mathrm{x}_\beta^\circ\kappa_1\varkappa_\alpha^2}{\varkappa_\alpha^2\varkappa_\beta^2(\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1)},\quad \varsigma_{13}=\frac{\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ}{(\mathrm{x}_\alpha^\circ\kappa_2+\mathrm{x}_\beta^\circ\kappa_1)}\left(\frac{1}{\varkappa_\alpha^2}-\frac{1}{\varkappa_\beta^2}\right),\quad \varsigma_{14}=\mathrm{x}_\alpha^\circ\mathrm{x}_\beta^\circ\left(\frac{1}{\varkappa_\alpha^2}-\frac{1}{\varkappa_\beta^2}\right),$$

$$\varsigma_{15}=\frac{\kappa_2\kappa_3\varkappa_\beta-\kappa_1\kappa_4\varkappa_\alpha}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)},\quad \varsigma_{16}=\frac{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\alpha+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\beta}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)},\quad \varsigma_{17}=\frac{\varkappa_\alpha\varkappa_\beta(\kappa_8\kappa_3\varkappa_\beta-\kappa_7\kappa_4\varkappa_\alpha)}{\mathrm{x}^\circ_\alpha\kappa_8\varkappa_\beta^2+\mathrm{x}^\circ_\beta\kappa_7\varkappa_\alpha^2},$$

$$\varsigma_{18}=\frac{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha}{\mathrm{x}^\circ_\alpha\kappa_8\varkappa_\beta^2+\mathrm{x}^\circ_\beta\kappa_7\varkappa_\alpha^2},\quad \varsigma_{19}=\frac{\varkappa_\alpha\varkappa_\beta(\kappa_{10}\kappa_3\varkappa_\beta-\kappa_9\kappa_4\varkappa_\alpha)}{\mathrm{x}^\circ_\alpha\kappa_{10}\varkappa_\beta^2+\mathrm{x}^\circ_\beta\kappa_9\varkappa_\alpha^2},\quad \varsigma_{20}=\frac{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha}{\mathrm{x}^\circ_\alpha\kappa_{10}\varkappa_\beta^2+\mathrm{x}^\circ_\beta\kappa_9\varkappa_\alpha^2},$$

$$\varsigma_{21}=\frac{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\alpha+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\beta}{\varkappa_\alpha\varkappa_\beta},\quad \varsigma_{22}=\frac{\varkappa_\alpha\varkappa_\beta(\kappa_5\kappa_8\varkappa_\beta-\kappa_6\kappa_7\varkappa_\alpha)}{\mathrm{x}^\circ_\alpha\kappa_8\varkappa_\beta^2+\mathrm{x}^\circ_\beta\kappa_7\varkappa_\alpha^2},\quad \varsigma_{23}=\frac{\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\alpha}{\mathrm{x}^\circ_\alpha\kappa_8\varkappa_\beta^2+\mathrm{x}^\circ_\beta\kappa_7\varkappa_\alpha^2},$$

$$\varsigma_{24}=\frac{\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\alpha+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\beta}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)},\quad \varsigma_{25}=\frac{\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\beta^3+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\alpha^3}{\varkappa_\alpha^2\varkappa_\beta^2(\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\alpha)},$$

$$\varsigma_{26}=\frac{\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta}{(\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\alpha)}\left(\frac{1}{\varkappa_\alpha^2}-\frac{1}{\varkappa_\beta^2}\right),\quad \varsigma_{27}=\frac{\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta}{(\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha)}\left(\frac{1}{\varkappa_\alpha^2}-\frac{1}{\varkappa_\beta^2}\right),$$

$$\varsigma_{28}=\frac{\kappa_6\kappa_7\varkappa_\beta-\kappa_5\kappa_8\varkappa_\alpha}{\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\alpha},\quad \varsigma_{29}=\frac{\mathrm{x}^\circ_\alpha\kappa_8+\mathrm{x}^\circ_\beta\kappa_7}{\mathrm{x}^\circ_\alpha\kappa_6\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_5\varkappa_\alpha},\quad \varsigma_{30}=\frac{\mathrm{x}^\circ_\alpha\kappa_8+\mathrm{x}^\circ_\beta\kappa_7}{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha},$$

$$\varsigma_{31}=\frac{\mathrm{x}^\circ_\alpha\kappa_{10}+\mathrm{x}^\circ_\beta\kappa_9}{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha},\quad \varsigma_{32}=\frac{5}{2}\mathrm{x}^\circ_\beta\mathfrak{a}_3(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa},\quad \varsigma_{33}=\frac{\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta^3+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha^3}{\varkappa_\alpha^2\varkappa_\beta^2(\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha)}.$$

The coefficients ϖ_i's are as follows.

$$\begin{aligned}\varpi_1&=\frac{\mathrm{x}^\circ_\alpha\varkappa_\beta\{\kappa_2(\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_3)-\kappa_1\mathrm{x}^\circ_\beta\delta_4\}+\mathrm{x}^\circ_\beta\varkappa_\alpha\{\kappa_1(\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_3)-\kappa_2\mathrm{x}^\circ_\alpha\gamma_4\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)}\\&=\frac{(\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_3)(\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_3)-\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta\delta_4\gamma_4}{\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1},\\\varpi_2&=\frac{\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta\big[\varkappa_\beta\{(\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_3)+\mathrm{x}^\circ_\alpha\delta_4\}-\varkappa_\alpha\{(\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_3)+\mathrm{x}^\circ_\beta\gamma_4\}\big]}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)}\\&=\frac{\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta\{(\kappa_1-\kappa_2)+\varkappa_\beta\delta_4-\varkappa_\alpha\gamma_4\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)},\end{aligned}$$

$$\varpi_3=\frac{\kappa_1\varkappa_\beta\{\kappa_2(\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_3)-\kappa_1\mathrm{x}^\circ_\beta\delta_4\}-\kappa_2\varkappa_\alpha\{\kappa_1(\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_3)-\kappa_2\mathrm{x}^\circ_\alpha\gamma_4\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)}=(\kappa_1-\kappa_2)\varpi_1$$

$$\begin{aligned}\varpi_4&=\frac{\kappa_1\varkappa_\beta\mathrm{x}^\circ_\beta\{(\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_3)+\mathrm{x}^\circ_\alpha\delta_4\}+\kappa_2\varkappa_\alpha\mathrm{x}^\circ_\alpha\{(\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_3)+\mathrm{x}^\circ_\beta\gamma_4\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)}\\&=\frac{\mathrm{x}^\circ_\alpha\kappa_2^2+\mathrm{x}^\circ_\beta\kappa_1^2+\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta(\kappa_1-\kappa_2)(\varkappa_\beta\delta_4-\varkappa_\alpha\gamma_4)}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_2+\mathrm{x}^\circ_\beta\kappa_1)},\end{aligned}$$

$$\begin{aligned}\varpi_5&=\frac{\mathrm{x}^\circ_\alpha\varkappa_\beta^2\left\{\kappa_4\left(\frac{2}{3}\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_5\right)-\kappa_3\mathrm{x}^\circ_\beta\delta_6\right\}+\mathrm{x}^\circ_\beta\varkappa_\alpha^2\left\{\kappa_3\left(\frac{2}{3}\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_5\right)-\kappa_4\mathrm{x}^\circ_\alpha\gamma_6\right\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha)}\\&=\frac{(\mathrm{x}^\circ_\alpha\varkappa_\beta^2+\mathrm{x}^\circ_\beta\varkappa_\alpha^2)\left\{\left(\frac{2}{3}\mathrm{x}^\circ_\alpha\Omega_\alpha+\mathrm{x}^\circ_\beta\delta_5\right)\left(\frac{2}{3}\mathrm{x}^\circ_\beta\Omega_\beta+\mathrm{x}^\circ_\alpha\gamma_5\right)-\mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta\delta_6\gamma_6\right\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_4\varkappa_\beta+\mathrm{x}^\circ_\beta\kappa_3\varkappa_\alpha)},\end{aligned}$$

$$\varpi_6 = \frac{\mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \left[\varkappa_\beta^2 \left\{\varkappa_\alpha \left(\frac{2}{3}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_5\right) + \varkappa_\beta \mathrm{x}_\alpha^\circ \delta_6\right\} - \varkappa_\alpha^2 \left\{\varkappa_\beta \left(\frac{2}{3}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_5\right) + \varkappa_\alpha \mathrm{x}_\beta^\circ \gamma_6\right\}\right]}{\varkappa_\alpha^2 \varkappa_\beta^2 (\mathrm{x}_\alpha^\circ \kappa_4 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_3 \varkappa_\alpha)}$$
$$= \frac{\mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \left\{\varkappa_\alpha \varkappa_\beta (\kappa_3 \varkappa_\beta - \kappa_4 \varkappa_\alpha) + (\mathrm{x}_\alpha^\circ \varkappa_\beta^2 + \mathrm{x}_\beta^\circ \varkappa_\alpha^2)(\varkappa_\beta \delta_6 - \varkappa_\alpha \gamma_6)\right\}}{\varkappa_\alpha^2 \varkappa_\beta^2 (\mathrm{x}_\alpha^\circ \kappa_4 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_3 \varkappa_\alpha)},$$

$$\varpi_7 = \frac{\kappa_3 \varkappa_\beta \left\{\kappa_4 \left(\frac{2}{3}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_5\right) - \kappa_3 \mathrm{x}_\beta^\circ \delta_6\right\} - \kappa_4 \varkappa_\alpha \left\{\kappa_3 \left(\frac{2}{3}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_5\right) - \kappa_4 \mathrm{x}_\alpha^\circ \gamma_6\right\}}{\mathrm{x}_\alpha^\circ \kappa_4 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_3 \varkappa_\alpha}$$
$$= \frac{(\kappa_3 \varkappa_\beta - \kappa_4 \varkappa_\alpha)\left\{\left(\frac{2}{3}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_5\right)\left(\frac{2}{3}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_5\right) - \mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \delta_6 \gamma_6\right\}}{\mathrm{x}_\alpha^\circ \kappa_4 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_3 \varkappa_\alpha}$$
$$= \frac{\varkappa_\alpha \varkappa_\beta (\kappa_3 \varkappa_\beta - \kappa_4 \varkappa_\alpha)}{(\mathrm{x}_\alpha^\circ \varkappa_\beta^2 + \mathrm{x}_\beta^\circ \varkappa_\alpha^2)} \varpi_5,$$

$$\varpi_8 = \frac{\kappa_3 \varkappa_\beta \mathrm{x}_\beta^\circ \left\{\varkappa_\alpha \left(\frac{2}{3}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_5\right) + \varkappa_\beta \mathrm{x}_\alpha^\circ \delta_6\right\} + \kappa_4 \varkappa_\alpha \mathrm{x}_\alpha^\circ \left\{\varkappa_\beta \left(\frac{2}{3}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_5\right) + \varkappa_\alpha \mathrm{x}_\beta^\circ \gamma_6\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_4 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_3 \varkappa_\alpha)}$$
$$= \frac{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_4^2 + \mathrm{x}_\beta^\circ \kappa_3^2) + \mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ (\kappa_3 \varkappa_\beta - \kappa_4 \varkappa_\alpha)(\varkappa_\beta \delta_6 - \varkappa_\alpha \gamma_6)}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_4 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_3 \varkappa_\alpha)},$$

$$\varpi_9 = \frac{\mathrm{x}_\alpha^\circ \varkappa_\beta^2 \left\{\kappa_6 \left(\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_7\right) - \kappa_5 \mathrm{x}_\beta^\circ \delta_8\right\} + \mathrm{x}_\beta^\circ \varkappa_\alpha^2 \left\{\kappa_5 \left(\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_7\right) - \kappa_6 \mathrm{x}_\alpha^\circ \gamma_8\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha)}$$
$$= \frac{(\mathrm{x}_\alpha^\circ \kappa_2 \varkappa_\beta^2 + \mathrm{x}_\beta^\circ \kappa_1 \varkappa_\alpha^2)\left\{\left(\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_7\right)\left(\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_7\right) - \mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \delta_8 \gamma_8\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha)},$$

$$\varpi_{10} = \frac{\mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \left[\varkappa_\beta^2 \left\{\varkappa_\alpha \left(\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_7\right) + \varkappa_\beta \mathrm{x}_\alpha^\circ \delta_8\right\} - \varkappa_\alpha^2 \left\{\varkappa_\beta \left(\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_7\right) + \varkappa_\alpha \mathrm{x}_\beta^\circ \gamma_8\right\}\right]}{\varkappa_\alpha^2 \varkappa_\beta^2 (\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha)}$$
$$= \frac{\mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \left\{\varkappa_\alpha \varkappa_\beta (\kappa_5 \kappa_2 \varkappa_\beta - \kappa_6 \kappa_1 \varkappa_\alpha) + (\mathrm{x}_\alpha^\circ \kappa_2 \varkappa_\beta^2 + \mathrm{x}_\beta^\circ \kappa_1 \varkappa_\alpha^2)(\kappa_1 \varkappa_\beta \delta_8 - \kappa_2 \varkappa_\alpha \gamma_8)\right\}}{\kappa_1 \kappa_2 \varkappa_\alpha^2 \varkappa_\beta^2 (\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha)},$$

$$\varpi_{11} = \frac{\kappa_5 \varkappa_\beta \left\{\kappa_6 \left(\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_7\right) - \kappa_5 \mathrm{x}_\beta^\circ \delta_8\right\} - \kappa_6 \varkappa_\alpha \left\{\kappa_5 \left(\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_7\right) - \kappa_6 \mathrm{x}_\alpha^\circ \gamma_8\right\}}{\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha}$$
$$= \frac{(\kappa_5 \kappa_2 \varkappa_\beta - \kappa_6 \kappa_1 \varkappa_\alpha)\left\{\left(\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_7\right)\left(\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_7\right) - \mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ \delta_8 \gamma_8\right\}}{\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha}$$
$$= \frac{\varkappa_\alpha \varkappa_\beta (\kappa_5 \kappa_2 \varkappa_\beta - \kappa_6 \kappa_1 \varkappa_\alpha)}{(\mathrm{x}_\alpha^\circ \kappa_2 \varkappa_\beta^2 + \mathrm{x}_\beta^\circ \kappa_1 \varkappa_\alpha^2)} \varpi_9,$$

$$\varpi_{12} = \frac{\kappa_5 \varkappa_\beta \mathrm{x}_\beta^\circ \left\{\varkappa_\alpha \left(\frac{3}{2}\mathrm{x}_\alpha^\circ \Omega_\alpha + \mathrm{x}_\beta^\circ \delta_7\right) + \varkappa_\beta \mathrm{x}_\alpha^\circ \delta_8\right\} + \kappa_6 \varkappa_\alpha \mathrm{x}_\alpha^\circ \left\{\varkappa_\beta \left(\frac{3}{2}\mathrm{x}_\beta^\circ \Omega_\beta + \mathrm{x}_\alpha^\circ \gamma_7\right) + \varkappa_\alpha \mathrm{x}_\beta^\circ \gamma_8\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha)}$$
$$= \frac{\varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_1 \kappa_6^2 + \mathrm{x}_\beta^\circ \kappa_2 \kappa_5^2) + \mathrm{x}_\alpha^\circ \mathrm{x}_\beta^\circ (\kappa_5 \kappa_2 \varkappa_\beta - \kappa_6 \kappa_1 \varkappa_\alpha)(\kappa_1 \varkappa_\beta \delta_8 - \kappa_2 \varkappa_\alpha \gamma_8)}{\kappa_1 \kappa_2 \varkappa_\alpha \varkappa_\beta (\mathrm{x}_\alpha^\circ \kappa_6 \varkappa_\beta + \mathrm{x}_\beta^\circ \kappa_5 \varkappa_\alpha)},$$

$$\varpi_{13} = \frac{\mathrm{x}^\circ_\alpha \varkappa_\beta^3 \left\{\kappa_8 \left(\frac{7}{6}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_9\right) - \kappa_7 \mathrm{x}^\circ_\beta \delta_{10}\right\} + \mathrm{x}^\circ_\beta \varkappa_\alpha^3 \left\{\kappa_7 \left(\frac{7}{6}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_9\right) - \kappa_8 \mathrm{x}^\circ_\alpha \gamma_{10}\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)}$$

$$= \frac{(\mathrm{x}^\circ_\alpha \kappa_4 \varkappa_\beta^3 + \mathrm{x}^\circ_\beta \kappa_3 \varkappa_\alpha^3) \left\{\left(\frac{7}{6}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_9\right)\left(\frac{7}{6}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_9\right) - \mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \delta_{10}\gamma_{10}\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)},$$

$$\varpi_{14} = \frac{\mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \left[\varkappa_\beta^3 \left\{\varkappa_\alpha^2 \left(\frac{7}{6}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_9\right) + \varkappa_\beta^2 \mathrm{x}^\circ_\alpha \delta_{10}\right\} - \varkappa_\alpha^3 \left\{\varkappa_\beta^2 \left(\frac{7}{6}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_9\right) + \varkappa_\alpha^2 \mathrm{x}^\circ_\beta \gamma_{10}\right\}\right]}{\varkappa_\alpha^3 \varkappa_\beta^3 (\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)}$$

$$= \frac{\mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \left\{\varkappa_\alpha^2 \varkappa_\beta^2 (\kappa_7 \kappa_4 \varkappa_\beta - \kappa_8 \kappa_3 \varkappa_\alpha) + (\mathrm{x}^\circ_\alpha \kappa_4 \varkappa_\beta^3 + \mathrm{x}^\circ_\beta \kappa_3 \varkappa_\alpha^3)(\kappa_3 \varkappa_\beta^2 \delta_{10} - \kappa_4 \varkappa_\alpha^2 \gamma_{10})\right\}}{\kappa_3 \kappa_4 \varkappa_\alpha^3 \varkappa_\beta^3 (\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)},$$

$$\varpi_{15} = \frac{\varkappa_\alpha \varkappa_\beta \left[\kappa_7 \varkappa_\beta \left\{\kappa_8 \left(\frac{7}{6}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_9\right) - \kappa_7 \mathrm{x}^\circ_\beta \delta_{10}\right\} - \kappa_8 \varkappa_\alpha \left\{\kappa_7 \left(\frac{7}{6}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_9\right) - \kappa_8 \mathrm{x}^\circ_\alpha \gamma_{10}\right\}\right]}{(\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)}$$

$$= \frac{\varkappa_\alpha \varkappa_\beta (\kappa_7 \kappa_4 \varkappa_\beta - \kappa_8 \kappa_3 \varkappa_\alpha) \left\{\left(\frac{7}{6}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_9\right)\left(\frac{7}{6}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_9\right) - \mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \delta_{10}\gamma_{10}\right\}}{(\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)}$$

$$= \frac{\varkappa_\alpha^2 \varkappa_\beta^2 (\kappa_7 \kappa_4 \varkappa_\beta - \kappa_8 \kappa_3 \varkappa_\alpha)}{(\mathrm{x}^\circ_\alpha \kappa_4 \varkappa_\beta^3 + \mathrm{x}^\circ_\beta \kappa_3 \varkappa_\alpha^3)} \varpi_{13},$$

$$\varpi_{16} = \frac{\kappa_7 \varkappa_\beta \mathrm{x}^\circ_\beta \left\{\varkappa_\alpha^2 \left(\frac{7}{6}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_9\right) + \varkappa_\beta^2 \mathrm{x}^\circ_\alpha \delta_{10}\right\} + \kappa_8 \varkappa_\alpha \mathrm{x}^\circ_\alpha \left\{\varkappa_\beta^2 \left(\frac{7}{6}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_9\right) + \varkappa_\alpha^2 \mathrm{x}^\circ_\beta \gamma_{10}\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)}$$

$$= \frac{\varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_3 \kappa_8^2 \varkappa_\beta + \mathrm{x}^\circ_\beta \kappa_4 \kappa_7^2 \varkappa_\alpha) + \mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta (\kappa_7 \kappa_4 \varkappa_\beta - \kappa_8 \kappa_3 \varkappa_\alpha)(\kappa_3 \varkappa_\beta^2 \delta_{10} - \kappa_4 \varkappa_\alpha^2 \gamma_{10})}{\kappa_3 \kappa_4 \varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_8 \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_7 \varkappa_\alpha^2)},$$

$$\varpi_{17} = \frac{\mathrm{x}^\circ_\alpha \varkappa_\beta^3 \left\{\kappa_{10} \left(\frac{2}{3}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_{11}\right) - \kappa_9 \mathrm{x}^\circ_\beta \delta_{12}\right\} + \mathrm{x}^\circ_\beta \varkappa_\alpha^3 \left\{\kappa_9 \left(\frac{2}{3}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_{11}\right) - \kappa_{10} \mathrm{x}^\circ_\alpha \gamma_{12}\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_{10} \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_9 \varkappa_\alpha^2)}$$

$$= \frac{(\mathrm{x}^\circ_\alpha \kappa_4 \varkappa_\beta^3 + \mathrm{x}^\circ_\beta \kappa_3 \varkappa_\alpha^3) \left\{\left(\frac{2}{3}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_{11}\right)\left(\frac{2}{3}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_{11}\right) - \mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \delta_{12}\gamma_{12}\right\}}{\varkappa_\alpha \varkappa_\beta (\mathrm{x}^\circ_\alpha \kappa_{10} \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_9 \varkappa_\alpha^2)},$$

$$\varpi_{18} = \frac{\mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \left[\varkappa_\beta^3 \left\{\varkappa_\alpha^2 \left(\frac{2}{3}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_{11}\right) + \varkappa_\beta^2 \mathrm{x}^\circ_\alpha \delta_{12}\right\} - \varkappa_\alpha^3 \left\{\varkappa_\beta^2 \left(\frac{2}{3}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_{11}\right) + \varkappa_\alpha^2 \mathrm{x}^\circ_\beta \gamma_{12}\right\}\right]}{\varkappa_\alpha^3 \varkappa_\beta^3 (\mathrm{x}^\circ_\alpha \kappa_{10} \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_9 \varkappa_\alpha^2)}$$

$$= \frac{\mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \left\{\varkappa_\alpha^2 \varkappa_\beta^2 (\kappa_9 \kappa_4 \varkappa_\beta - \kappa_{10} \kappa_3 \varkappa_\alpha) + (\mathrm{x}^\circ_\alpha \kappa_4 \varkappa_\beta^3 + \mathrm{x}^\circ_\beta \kappa_3 \varkappa_\alpha^3)(\kappa_3 \varkappa_\beta^2 \delta_{12} - \kappa_4 \varkappa_\alpha^2 \gamma_{12})\right\}}{\kappa_3 \kappa_4 \varkappa_\alpha^3 \varkappa_\beta^3 (\mathrm{x}^\circ_\alpha \kappa_{10} \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_9 \varkappa_\alpha^2)},$$

$$\varpi_{19} = \frac{\varkappa_\alpha \varkappa_\beta \left[\kappa_9 \varkappa_\beta \left\{\kappa_{10} \left(\frac{2}{3}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_{11}\right) - \kappa_9 \mathrm{x}^\circ_\beta \delta_{12}\right\} - \kappa_{10} \varkappa_\alpha \left\{\kappa_9 \left(\frac{2}{3}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_{11}\right) - \kappa_{10} \mathrm{x}^\circ_\alpha \gamma_{12}\right\}\right]}{(\mathrm{x}^\circ_\alpha \kappa_{10} \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_9 \varkappa_\alpha^2)}$$

$$= \frac{\varkappa_\alpha \varkappa_\beta (\kappa_9 \kappa_4 \varkappa_\beta - \kappa_{10} \kappa_3 \varkappa_\alpha) \left\{\left(\frac{2}{3}\mathrm{x}^\circ_\alpha \Omega_\alpha + \mathrm{x}^\circ_\beta \delta_{11}\right)\left(\frac{2}{3}\mathrm{x}^\circ_\beta \Omega_\beta + \mathrm{x}^\circ_\alpha \gamma_{11}\right) - \mathrm{x}^\circ_\alpha \mathrm{x}^\circ_\beta \delta_{12}\gamma_{12}\right\}}{(\mathrm{x}^\circ_\alpha \kappa_{10} \varkappa_\beta^2 + \mathrm{x}^\circ_\beta \kappa_9 \varkappa_\alpha^2)}$$

$$= \frac{\varkappa_\alpha^2 \varkappa_\beta^2 (\kappa_9 \kappa_4 \varkappa_\beta - \kappa_{10} \kappa_3 \varkappa_\alpha)}{(\mathrm{x}^\circ_\alpha \kappa_4 \varkappa_\beta^3 + \mathrm{x}^\circ_\beta \kappa_3 \varkappa_\alpha^3)} \varpi_{17},$$

$$\varpi_{20} = \frac{\kappa_9\varkappa_\beta \mathrm{x}^\circ_\beta \left\{\varkappa_\alpha^2\left(\frac{2}{3}\mathrm{x}^\circ_\alpha\Omega_\alpha + \mathrm{x}^\circ_\beta\delta_{11}\right) + \varkappa_\beta^2\mathrm{x}^\circ_\alpha\delta_{12}\right\} + \kappa_{10}\varkappa_\alpha\mathrm{x}^\circ_\alpha\left\{\varkappa_\beta^2\left(\frac{2}{3}\mathrm{x}^\circ_\beta\Omega_\beta + \mathrm{x}^\circ_\alpha\gamma_{11}\right) + \varkappa_\alpha^2\mathrm{x}^\circ_\beta\gamma_{12}\right\}}{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_{10}\varkappa_\beta^2 + \mathrm{x}^\circ_\beta\kappa_9\varkappa_\alpha^2)}$$
$$= \frac{\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_3\kappa_{10}^2\varkappa_\beta + \mathrm{x}^\circ_\beta\kappa_4\kappa_9^2\varkappa_\alpha) + \mathrm{x}^\circ_\alpha\mathrm{x}^\circ_\beta(\kappa_9\kappa_4\varkappa_\beta - \kappa_{10}\kappa_3\varkappa_\alpha)(\kappa_3\varkappa_\beta^2\delta_{12} - \kappa_4\varkappa_\alpha^2\gamma_{12})}{\kappa_3\kappa_4\varkappa_\alpha\varkappa_\beta(\mathrm{x}^\circ_\alpha\kappa_{10}\varkappa_\beta^2 + \mathrm{x}^\circ_\beta\kappa_9\varkappa_\alpha^2)}.$$

The coefficients $\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{20}$ are as follows.

$$\mathfrak{a}_0 = \varsigma_2, \qquad \mathfrak{a}_1 = \delta_1\frac{\varkappa}{\varkappa_\beta^2} \qquad \mathfrak{a}_2 = \frac{4}{5}\left(1 - \frac{\varpi_2}{\varpi_4}\varsigma_9\right),$$
$$\mathfrak{a}_3 = 2\left(\varsigma_1 - \frac{\varpi_2}{\varpi_4}\varsigma_{11}\right), \qquad \mathfrak{a}_4 = \frac{1}{\varpi_4}(\varpi_1\varpi_4 - \varpi_2\varpi_3), \qquad \mathfrak{a}_5 = \varsigma_{12} - \frac{\varpi_6}{\varpi_8}\varsigma_{15},$$
$$\mathfrak{a}_6 = \frac{1}{\varpi_8}(\varpi_5\varpi_8 - \varpi_6\varpi_7), \qquad \mathfrak{a}_7 = \frac{\varpi_3}{\varpi_4}\varsigma_3, \qquad \mathfrak{a}_8 = 2\,\Omega\frac{\varsigma_3\varsigma_{11}}{\varpi_4},$$
$$\mathfrak{a}_9 = \frac{4}{5}\Omega\frac{\varsigma_3\varsigma_9}{\varpi_4}, \qquad \mathfrak{a}_{10} = \Omega\,\mathrm{x}^\circ_\beta\varkappa_\alpha\frac{\varsigma_6}{\delta_2}, \qquad \mathfrak{a}_{11} = \Omega\,\mathrm{x}^\circ_\beta\varkappa_\alpha\frac{\varsigma_4}{\delta_2},$$

$$\mathfrak{a}_{12} = \frac{\varpi_2}{\varpi_4}\left[\frac{\mathfrak{a}_6}{\varpi_4}\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\} - \frac{4}{5}\frac{\varpi_7}{\varpi_8}\varsigma_{10}\right],$$
$$\mathfrak{a}_{13} = 2\delta_1\frac{\varkappa}{\varkappa_\alpha\varkappa_\beta^2}\frac{\varsigma_{11}\varpi_2}{\varpi_4^2},$$
$$\mathfrak{a}_{14} = \Omega\frac{\varpi_2}{\varpi_4}\left[2\frac{\varsigma_2\varsigma_{11}}{\varpi_4}\frac{1}{\varkappa_\alpha} + \frac{\mathfrak{a}_5}{\varpi_4}\left\{\frac{4}{5}\varsigma_9 + \frac{\varpi_3}{\mathfrak{a}_4}\frac{1}{\varkappa}\left(5\frac{\eta}{\kappa} - \mathfrak{a}_2\varkappa + \frac{\varsigma_{32}}{\kappa}\right)\right\} - \frac{4}{5}\frac{\varsigma_{10}\varsigma_{15}}{\varpi_8}\right] + \frac{2}{5}\left(1 - \frac{\varpi_2}{\varpi_4}\varsigma_7\right)\zeta_m,$$
$$\mathfrak{a}_{15} = \frac{1}{3}\left(1 - \frac{\varpi_2}{\varpi_4}\varsigma_7\right)\zeta_m,$$
$$\mathfrak{a}_{16} = \mathfrak{a}_4\frac{\varpi_6}{\varpi_8^2}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} + \frac{\varpi_3}{\varpi_4}\left(\varsigma_{13} - \frac{\varpi_6}{\varpi_8}\varsigma_{16}\right),$$
$$\mathfrak{a}_{17} = \Omega\left[\mathfrak{a}_2\frac{\varpi_6}{\varpi_8^2}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} + \frac{4}{5}\frac{\varsigma_9}{\varpi_4}\left(\varsigma_{13} - \frac{\varpi_6}{\varpi_8}\varsigma_{16}\right)\right] + \frac{1}{2}\left(1 - \frac{\varpi_6}{\varpi_8}\varsigma_{17}\right)\zeta_R,$$
$$\mathfrak{a}_{18} = \Omega\left[\mathfrak{a}_3\frac{\varpi_6}{\varpi_8^2}\left\{\varsigma_{15} - \frac{\varpi_7}{\mathfrak{a}_6}\left(\mathfrak{a}_5 - \frac{\kappa}{2\eta}\right)\right\} + 2\frac{\varsigma_{11}}{\varpi_4}\left(\varsigma_{13} - \frac{\varpi_6}{\varpi_8}\varsigma_{16}\right)\right],$$
$$\mathfrak{a}_{19} = \frac{5}{2}\Omega\,\varkappa_\alpha\frac{\varsigma_4}{\delta_2}\left(\varsigma_{14} - \frac{\varpi_6}{\varpi_8}\varsigma_{21}\right) + \frac{1}{6}\left(1 - \frac{\varpi_6}{\varpi_8}\varsigma_{19}\right)\zeta_\Delta,$$
$$\mathfrak{a}_{20} = \frac{5}{2}\Omega\,\varkappa_\alpha\frac{\varsigma_6}{\delta_2}\left(\varsigma_{14} - \frac{\varpi_6}{\varpi_8}\varsigma_{21}\right).$$

Appendix D: Burnett equations: second order contributions to $\hat{\sigma}_{ij}$ and $\hat{h}_i$

To obtain the precise values of second order contributions to $\hat{\sigma}_{ij}$ and $\hat{h}_i$, let us perform the Chapman-Enskog like expansion on the new system of moment equations (48)–(59). We again expand the non-conserved quantities (Ψ) in powers of the Knudsen number (ε) as

$$\Psi = \Psi_{|0} + \varepsilon\,\Psi_{|1} + \varepsilon^2\Psi_{|2} + \ldots,$$

where $\Psi \in \{\hat{u}_i^{(\alpha)}, \Delta\hat{T}, \hat{\sigma}_{ij}, \Delta\hat{\sigma}_{ij}, \hat{h}_i, \Delta\hat{h}_i, \hat{m}_{ijk}, \Delta\hat{m}_{ijk}, \hat{R}_{ij}, \Delta\hat{R}_{ij}, \hat{\Delta}, \Delta\hat{\Delta}\}$ and the quantities $\Psi_{|0}$, $\Psi_{|1}$, $\Psi_{|2}$, ... are of order $\mathcal{O}(\varepsilon^0)$. Now, we shall substitute the above expansions in the new system of moment equations (48)–(59) and compare the coefficients of each power of ε.

Comparing coefficients of ε^{-1} on both sides of Eqs. (48)–(59), it immediately follows that $\Psi_{|0} = 0$ for all $\Psi \in \{\hat{u}_i^{(\alpha)}, \Delta\hat{T}, \hat{\sigma}_{ij}, \Delta\hat{\sigma}_{ij}, \hat{h}_i, \Delta\hat{h}_i, \hat{m}_{ijk}, \Delta\hat{m}_{ijk}, \hat{R}_{ij}, \Delta\hat{R}_{ij}, \hat{\Delta}, \Delta\hat{\Delta}\}$.

Comparing coefficients of ε^0 on both sides of Eqs. (48)–(59), it follows that

$$\left.\begin{aligned} \hat{u}_{i|1}^{(\alpha)} &= -\mathrm{x}_\beta^\circ \frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{x}_i} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{x}_i} - (\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{x}_i}\right), \\ \hat{\sigma}_{ij|1} &= -2\eta\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}, \qquad \hat{h}_{i|1} = -\kappa\frac{\partial\hat{T}}{\partial\hat{x}_i}, \\ \Delta\hat{T}_{|1} = \Delta\hat{\sigma}_{ij|1} &= \Delta\hat{h}_{i|1} = \hat{m}_{ijk|1} = \hat{R}_{ij|1} = \hat{\Delta}_{|1} = \Delta\hat{m}_{ijk|1} = \Delta\hat{R}_{ij|1} = \Delta\hat{\Delta}_{|1} = 0. \end{aligned}\right\} \tag{D1}$$

Comparing coefficients of ε^1 on both sides of Eqs. (48)–(59), it follows that

$$\begin{aligned} \hat{u}_{i|2}^{(\alpha)} &= -\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa}\left(\varkappa_\alpha\frac{\partial\hat{u}_{i|1}^{(\alpha)}}{\partial\hat{t}} + \varsigma_2\frac{\partial\hat{\sigma}_{ij|1}}{\partial\hat{x}_j}\right) \\ &= \frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa}\left[\mathrm{x}_\beta^\circ\frac{\Omega}{\delta_1}\frac{\varkappa_\alpha\varkappa_\beta^2}{\varkappa^2}\frac{\partial}{\partial\hat{x}_i}\left(\varkappa_\beta^2\frac{\partial\hat{n}_\alpha}{\partial\hat{t}} - \varkappa_\alpha^2\frac{\partial\hat{n}_\beta}{\partial\hat{t}} - (\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial\hat{T}}{\partial\hat{t}}\right) + 2\eta\varsigma_2\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right] \\ &= \frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa}\left[\frac{5}{3}\mathrm{x}_\beta^\circ\frac{\Omega}{\delta_1}\frac{\varkappa_\alpha\varkappa_\beta^2}{\varkappa^2}(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\partial}{\partial\hat{x}_i}\frac{\partial\hat{v}_j}{\partial\hat{x}_j} + 2\eta\varsigma_2\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}\right], \end{aligned} \tag{D2}$$

$$\begin{aligned} \Delta\hat{T}_{|2} &= -\Omega\frac{\varkappa_\alpha}{\delta_2}\left(\varsigma_4\frac{\partial\hat{h}_{i|1}}{\partial\hat{x}_i} + \varsigma_6\frac{\partial\hat{u}_{i|1}^{(\alpha)}}{\partial\hat{x}_i}\right) \\ &= \Omega\frac{\varkappa_\alpha}{\delta_2}\left[\varsigma_4\kappa\frac{\partial^2\hat{T}}{\partial\hat{x}_i^2} + \varsigma_6\mathrm{x}_\beta^\circ\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\frac{\partial^2}{\partial\hat{x}_i^2}\left(\varkappa_\beta^2\hat{n}_\alpha - \varkappa_\alpha^2\hat{n}_\beta - (\varkappa_\alpha^2-\varkappa_\beta^2)\hat{T}\right)\right], \end{aligned} \tag{D3}$$

$$\left.\begin{aligned} \frac{\partial\hat{\sigma}_{ij|1}}{\partial\hat{t}} + \frac{4}{5}\frac{\partial\hat{h}_{\langle i|1}}{\partial\hat{x}_{j\rangle}} + 2\varsigma_1\frac{\partial\hat{u}_{\langle i|1}^{(\alpha)}}{\partial\hat{x}_{j\rangle}} &= -\frac{1}{\Omega}\left(\varpi_1\hat{\sigma}_{ij|2} + \varpi_2\Delta\hat{\sigma}_{ij|2}\right), \\ \frac{4}{5}\varsigma_9\frac{\partial\hat{h}_{\langle i|1}}{\partial\hat{x}_{j\rangle}} + 2\varsigma_{11}\frac{\partial\hat{u}_{\langle i|1}^{(\alpha)}}{\partial\hat{x}_{j\rangle}} &= -\frac{1}{\Omega}\left(\varpi_3\hat{\sigma}_{ij|2} + \varpi_4\Delta\hat{\sigma}_{ij|2}\right), \end{aligned}\right\} \tag{D4}$$

$$\left.\begin{aligned} \frac{\partial\hat{h}_{i|1}}{\partial\hat{t}} + \varsigma_{12}\frac{\partial\hat{\sigma}_{ij|1}}{\partial\hat{x}_j} &= -\frac{1}{\Omega}\left(\varpi_5\hat{h}_{i|2} + \varpi_6\Delta\hat{h}_{i|2}\right), \\ \varsigma_{15}\frac{\partial\hat{\sigma}_{ij|1}}{\partial\hat{x}_j} &= -\frac{1}{\Omega}\left(\varpi_7\hat{h}_{i|2} + \varpi_8\Delta\hat{h}_{i|2}\right), \end{aligned}\right\} \tag{D5}$$

$$\hat{m}_{ijk|2} = -\zeta_m \frac{\partial \hat{\sigma}_{\langle ij|1}}{\partial \hat{x}_{k\rangle}}, \quad \hat{R}_{ij|2} = -\zeta_R \frac{\partial \hat{h}_{\langle i|1}}{\partial \hat{x}_{j\rangle}}, \quad \hat{\Delta}_{|2} = -\zeta_\Delta \frac{\partial \hat{h}_i}{\partial \hat{x}_i} \tag{D6}$$

and

$$\Delta\hat{m}_{ijk|2} = \Delta\hat{R}_{ij|2} = \Delta\hat{\Delta}_{|2} = 0. \tag{D7}$$

From Eqs. (D4) and (D5), we have

$$\left.\begin{aligned}
\hat{\sigma}_{ij|2} &= -\frac{\Omega}{(\varpi_1\varpi_4 - \varpi_2\varpi_3)}\left[\varpi_4 \frac{\partial \hat{\sigma}_{ij|1}}{\partial \hat{t}} + \frac{4}{5}(\varpi_4 - \varpi_2\varsigma_9)\frac{\partial \hat{h}_{\langle i|1}}{\partial \hat{x}_{j\rangle}} + 2(\varpi_4\varsigma_1 - \varpi_2\varsigma_{11})\frac{\partial \hat{u}^{(\alpha)}_{\langle i|1}}{\partial \hat{x}_{j\rangle}}\right], \\
\Delta\hat{\sigma}_{ij|2} &= \frac{\Omega}{(\varpi_1\varpi_4 - \varpi_2\varpi_3)}\left[\varpi_3 \frac{\partial \hat{\sigma}_{ij|1}}{\partial \hat{t}} + \frac{4}{5}(\varpi_3 - \varpi_1\varsigma_9)\frac{\partial \hat{h}_{\langle i|1}}{\partial \hat{x}_{j\rangle}} + 2(\varpi_3\varsigma_1 - \varpi_1\varsigma_{11})\frac{\partial \hat{u}^{(\alpha)}_{\langle i|1}}{\partial \hat{x}_{j\rangle}}\right],
\end{aligned}\right\} \tag{D8}$$

$$\left.\begin{aligned}
\hat{h}_{i|2} &= -\frac{\Omega}{(\varpi_5\varpi_8 - \varpi_6\varpi_7)}\left[\varpi_8 \frac{\partial \hat{h}_{i|1}}{\partial \hat{t}} + (\varpi_8\varsigma_{12} - \varpi_6\varsigma_{15})\frac{\partial \hat{\sigma}_{ij|1}}{\partial \hat{x}_j}\right], \\
\Delta\hat{h}_{i|2} &= \frac{\Omega}{(\varpi_5\varpi_8 - \varpi_6\varpi_7)}\left[\varpi_7 \frac{\partial \hat{h}_{i|1}}{\partial \hat{t}} + (\varpi_7\varsigma_{12} - \varpi_5\varsigma_{15})\frac{\partial \hat{\sigma}_{ij|1}}{\partial \hat{x}_j}\right].
\end{aligned}\right\} \tag{D9}$$

From Eqs. $(D1)_{2,3}$ on using momentum and energy balance equations (Eqs. (46) and (47)) with $\hat{u}_i^{(\alpha)} = \hat{\sigma}_{ij} = \hat{q}_i = 0$ (i.e., on using the Euler equations), we have

$$\frac{\partial \hat{\sigma}_{ij|1}}{\partial \hat{t}} = -2\eta \frac{\partial}{\partial \hat{x}_{j\rangle}}\frac{\partial \hat{v}_{\langle i}}{\partial \hat{t}} = 2\frac{\eta}{\varkappa}\frac{\partial^2}{\partial \hat{x}_{\langle i}\partial \hat{x}_{j\rangle}}\left(\mathrm{x}^\circ_\alpha \hat{n}_\alpha + \mathrm{x}^\circ_\beta \hat{n}_\beta + \hat{T}\right), \tag{D10}$$

$$\frac{\partial \hat{h}_{i|1}}{\partial \hat{t}} = -\kappa \frac{\partial}{\partial \hat{x}_i}\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{2}{3}\kappa \frac{\partial^2 \hat{v}_j}{\partial \hat{x}_i \partial \hat{x}_j}. \tag{D11}$$

Therefore, Eqs. (D8) and (D9) on using (D1) yield

$$\hat{\sigma}_{ij|2} = -\frac{\partial^2}{\partial \hat{x}_{\langle i}\hat{x}_{j\rangle}}\left(\mathfrak{b}_1 \hat{n}_\alpha + \mathfrak{b}_2 \hat{n}_\beta + \mathfrak{b}_3 \hat{T}\right), \tag{D12}$$

$$\Delta\hat{\sigma}_{ij|2} = \frac{\partial^2}{\partial \hat{x}_{\langle i}\partial \hat{x}_{j\rangle}}\left(\mathfrak{b}_4 \hat{n}_\alpha + \mathfrak{b}_5 \hat{n}_\beta + \mathfrak{b}_6 \hat{T}\right), \tag{D13}$$

$$\hat{h}_{i|2} = -\mathfrak{b}_7 \frac{\partial^2 \hat{v}_j}{\partial \hat{x}_i \partial \hat{x}_j} - 2\eta\mathfrak{b}_8 \frac{\partial}{\partial \hat{x}_j}\frac{\partial \hat{v}_{\langle i}}{\partial \hat{x}_{j\rangle}}, \tag{D14}$$

$$\Delta\hat{h}_{i|2} = \mathfrak{b}_9\frac{\partial^2\hat{v}_j}{\partial\hat{x}_i\partial\hat{x}_j} - 2\eta\mathfrak{b}_{10}\frac{\partial}{\partial\hat{x}_j}\frac{\partial\hat{v}_{\langle i}}{\partial\hat{x}_{j\rangle}}, \tag{D15}$$

where

$$\begin{aligned}
\mathfrak{b}_1 &= \frac{2\,\Omega}{(\varpi_1\varpi_4-\varpi_2\varpi_3)}\left\{\mathrm{x}^\circ_\alpha\varpi_4\frac{\eta}{\varkappa} - \mathrm{x}^\circ_\beta(\varpi_4\varsigma_1-\varpi_2\varsigma_{11})\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^4}{\varkappa^2}\right\} = \frac{\Omega}{\mathfrak{a}_4}\left\{\mathrm{x}^\circ_\alpha\frac{2\eta}{\varkappa} - \mathrm{x}^\circ_\beta\mathfrak{a}_3\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^4}{\varkappa^2}\right\},\\
\mathfrak{b}_2 &= \frac{2\,\Omega}{(\varpi_1\varpi_4-\varpi_2\varpi_3)}\mathrm{x}^\circ_\beta\left\{\varpi_4\frac{\eta}{\varkappa} + (\varpi_4\varsigma_1-\varpi_2\varsigma_{11})\frac{\Omega}{\delta_1}\frac{\varkappa_\alpha^2\varkappa_\beta^2}{\varkappa^2}\right\} = \frac{\Omega}{\mathfrak{a}_4}\mathrm{x}^\circ_\beta\left\{\frac{2\eta}{\varkappa} + \mathfrak{a}_3\frac{\Omega}{\delta_1}\frac{\varkappa_\alpha^2\varkappa_\beta^2}{\varkappa^2}\right\},\\
\mathfrak{b}_3 &= \frac{2\,\Omega}{(\varpi_1\varpi_4-\varpi_2\varpi_3)}\left\{\varpi_4\frac{\eta}{\varkappa} - \frac{2}{5}\kappa(\varpi_4-\varpi_2\varsigma_9) + \mathrm{x}^\circ_\beta(\varpi_4\varsigma_1-\varpi_2\varsigma_{11})(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\right\}\\
&= \frac{\Omega}{\mathfrak{a}_4}\left\{\frac{2\eta}{\varkappa} - \mathfrak{a}_2\kappa + \mathrm{x}^\circ_\beta\mathfrak{a}_3(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\right\},\\
\mathfrak{b}_4 &= \frac{2\,\Omega}{(\varpi_1\varpi_4-\varpi_2\varpi_3)}\left\{\mathrm{x}^\circ_\alpha\varpi_3\frac{\eta}{\varkappa} - \mathrm{x}^\circ_\beta(\varpi_3\varsigma_1-\varpi_1\varsigma_{11})\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^4}{\varkappa^2}\right\},\\
\mathfrak{b}_5 &= \frac{2\,\Omega}{(\varpi_1\varpi_4-\varpi_2\varpi_3)}\mathrm{x}^\circ_\beta\left\{\varpi_3\frac{\eta}{\varkappa} + (\varpi_3\varsigma_1-\varpi_1\varsigma_{11})\frac{\Omega}{\delta_1}\frac{\varkappa_\alpha^2\varkappa_\beta^2}{\varkappa^2}\right\},\\
\mathfrak{b}_6 &= \frac{2\,\Omega}{(\varpi_1\varpi_4-\varpi_2\varpi_3)}\left\{\varpi_3\frac{\eta}{\varkappa} - \frac{2}{5}\kappa(\varpi_3-\varpi_1\varsigma_9) + \mathrm{x}^\circ_\beta(\varpi_3\varsigma_1-\varpi_1\varsigma_{11})(\varkappa_\alpha^2-\varkappa_\beta^2)\frac{\Omega}{\delta_1}\frac{\varkappa_\beta^2}{\varkappa^2}\right\},\\
\mathfrak{b}_7 &= \frac{2}{3}\frac{\Omega}{(\varpi_5\varpi_8-\varpi_6\varpi_7)}\varpi_8\kappa = \frac{2}{3}\frac{\Omega}{\mathfrak{a}_6}\kappa,\\
\mathfrak{b}_8 &= -\frac{\Omega}{(\varpi_5\varpi_8-\varpi_6\varpi_7)}(\varpi_8\varsigma_{12}-\varpi_6\varsigma_{15}) = -\frac{\Omega\,\mathfrak{a}_5}{\mathfrak{a}_6},\\
\mathfrak{b}_9 &= \frac{2}{3}\frac{\Omega}{(\varpi_5\varpi_8-\varpi_6\varpi_7)}\varpi_7\kappa,\\
\mathfrak{b}_{10} &= \frac{\Omega}{(\varpi_5\varpi_8-\varpi_6\varpi_7)}(\varpi_7\varsigma_{12}-\varpi_5\varsigma_{15}).
\end{aligned}$$

Therefore, if the conservation laws for mixture are closed with up to second order corrections in the diffusion velocity (of one constituent), total stress and total reduced heat flux, i.e., with

$$\hat{u}_i^{(\alpha)} = \varepsilon\hat{u}_{i|1}^{(\alpha)} + \varepsilon^2\hat{u}_{i|2}^{(\alpha)}, \qquad \hat{\sigma}_{ij} = \varepsilon\hat{\sigma}_{ij|1} + \varepsilon^2\hat{\sigma}_{ij|2}, \qquad \hat{h}_i = \varepsilon\hat{h}_{i|1} + \varepsilon^2\hat{h}_{i|2}, \tag{D16}$$

where $\hat{u}_{i|1}^{(\alpha)}$, $\hat{\sigma}_{ij|1}$, $\hat{h}_{i|1}$, $\hat{u}_{i|2}^{(\alpha)}$, $\hat{\sigma}_{ij|2}$ and $\hat{h}_{i|2}$ are given by Eqs. (D1)$_{1,2,3}$, (D2), (D12) (D14), we essentially get the (linear) Burnett order equations for a binary gas mixture made up of Maxwell molecules.

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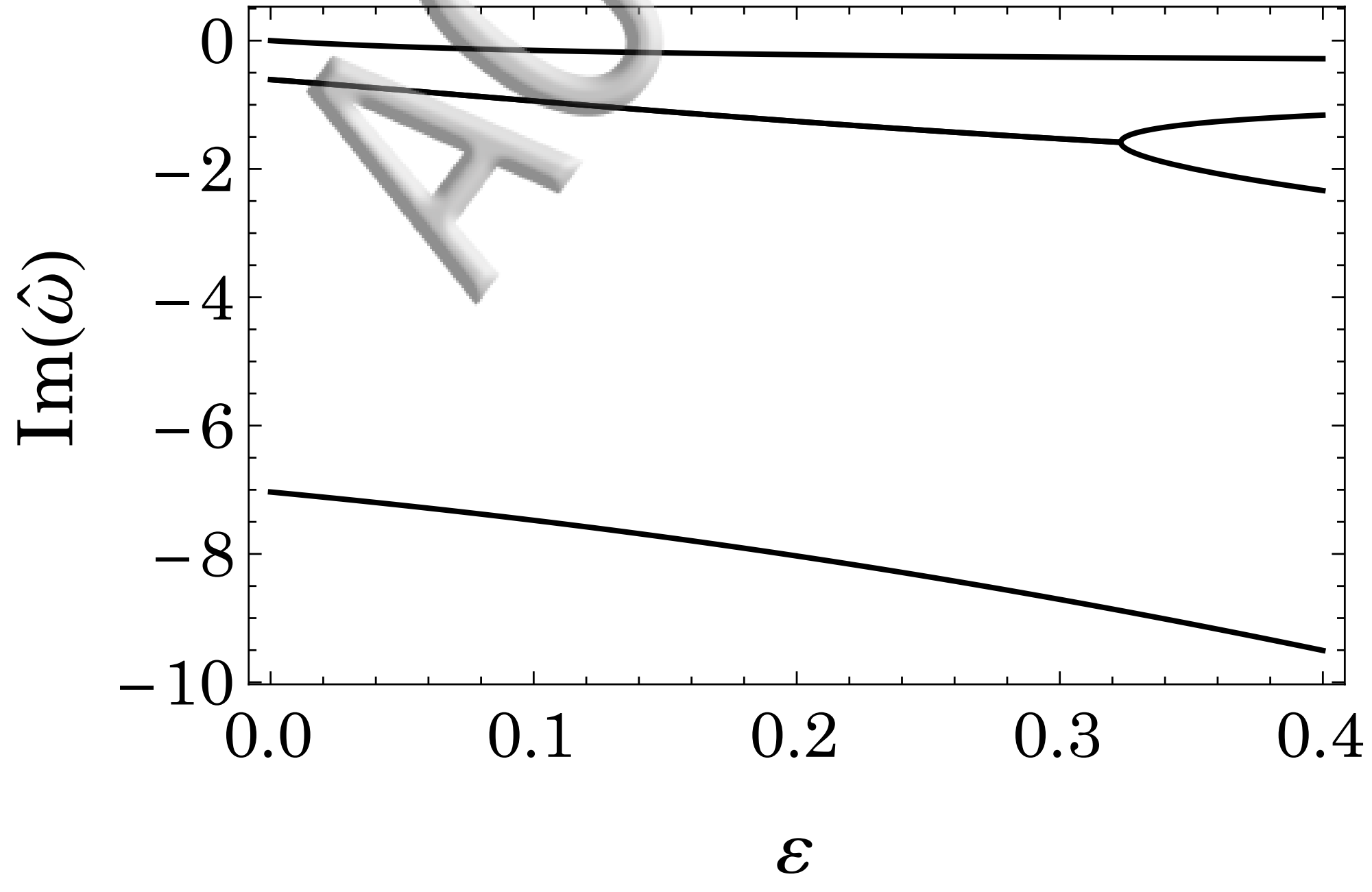
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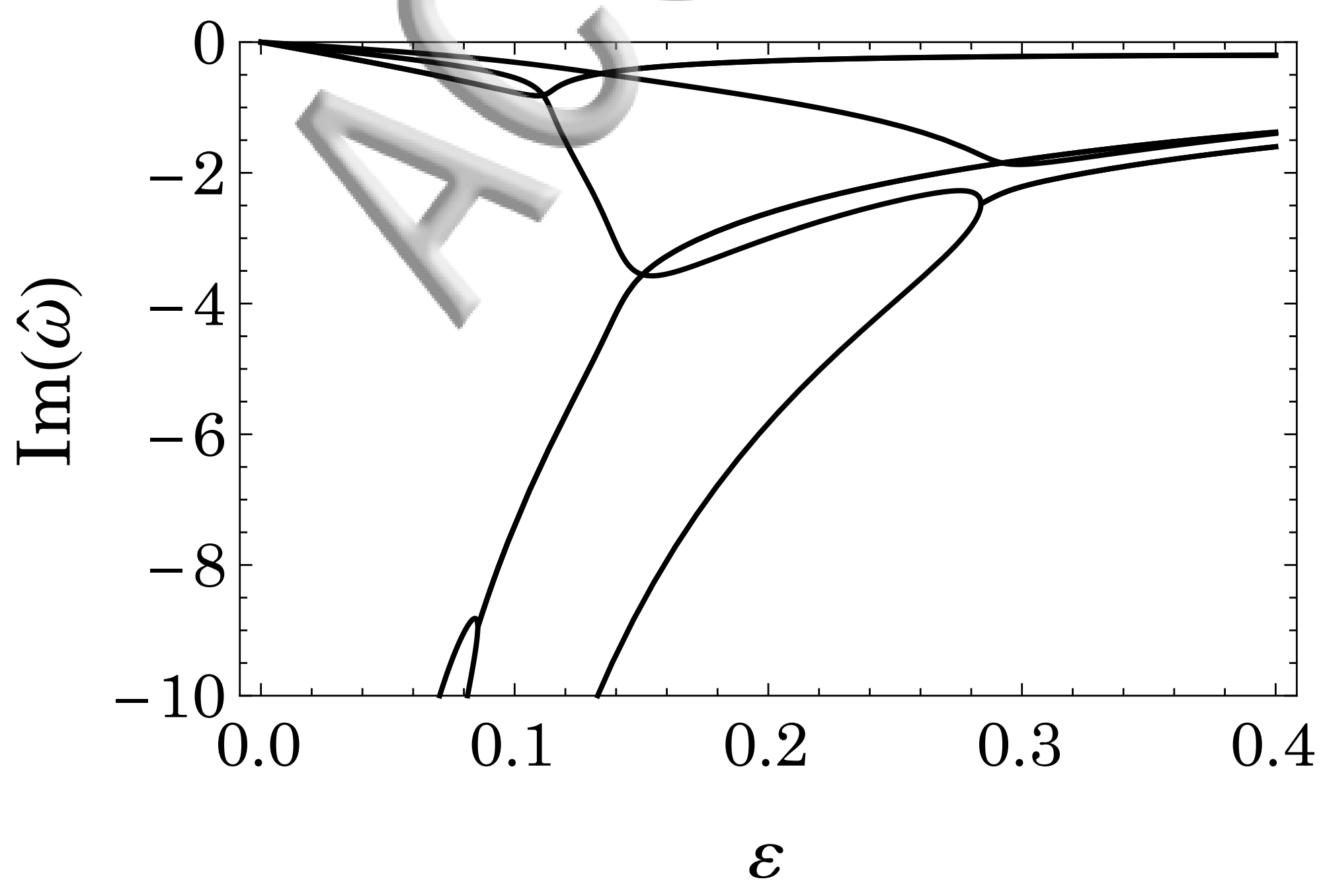
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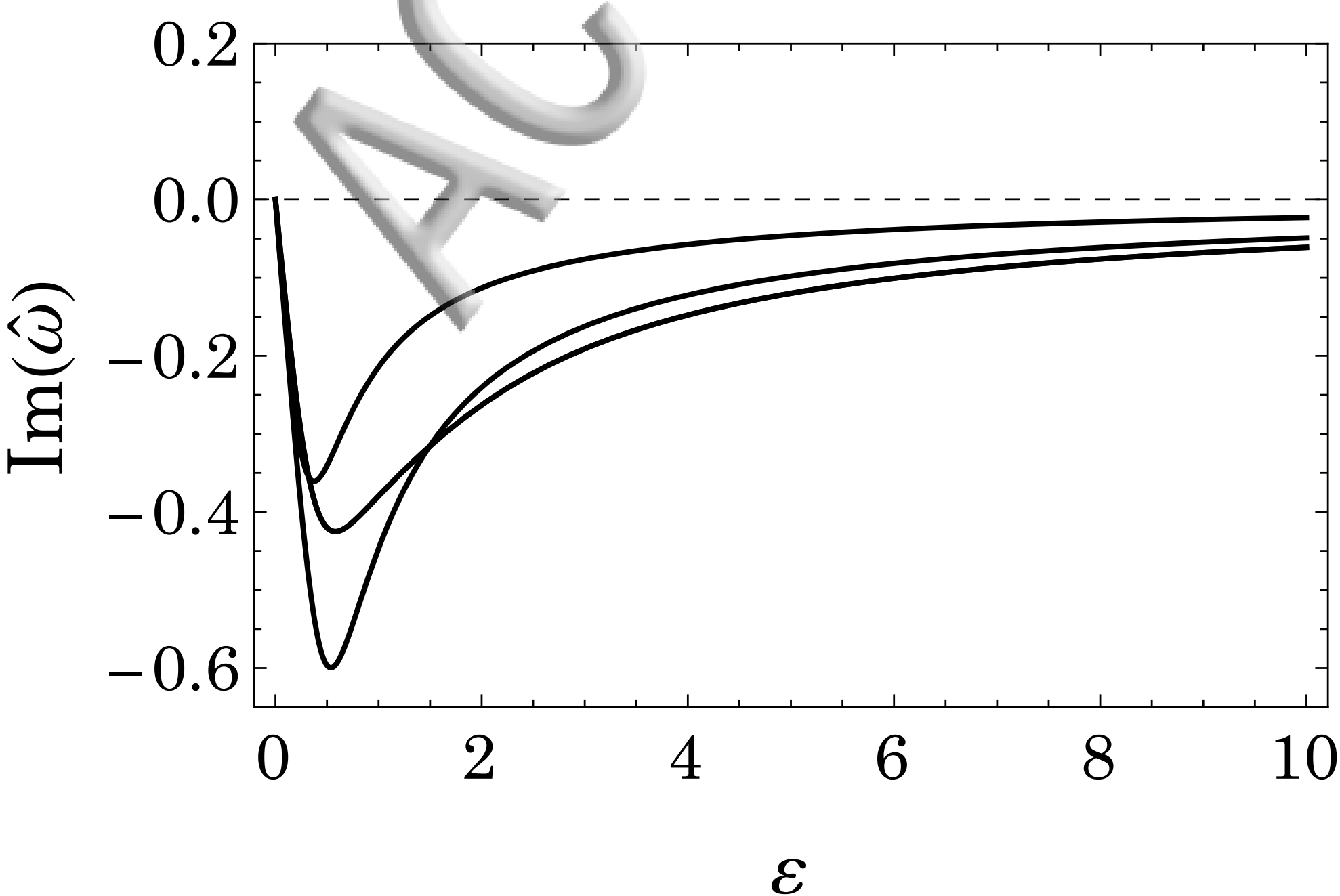
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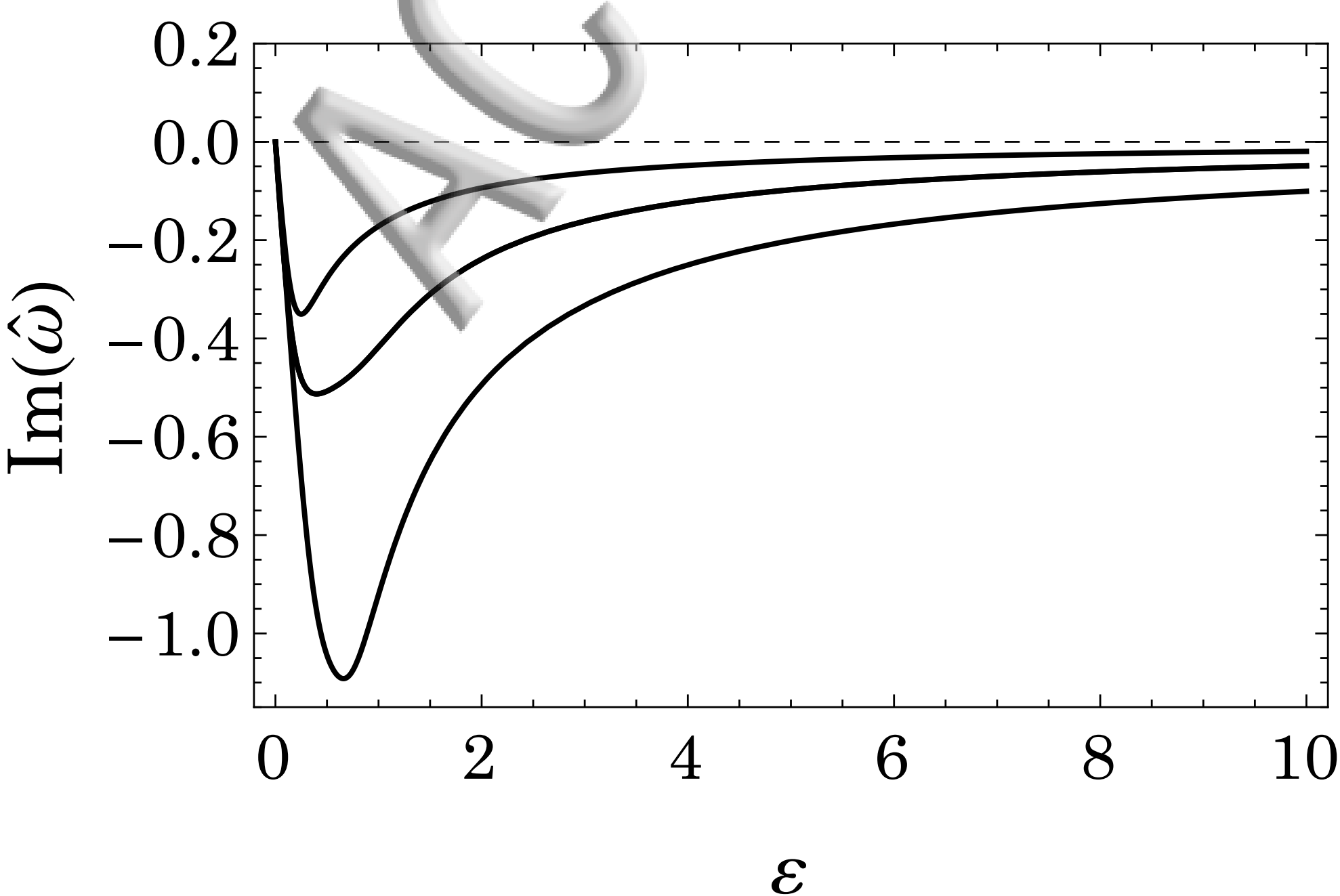
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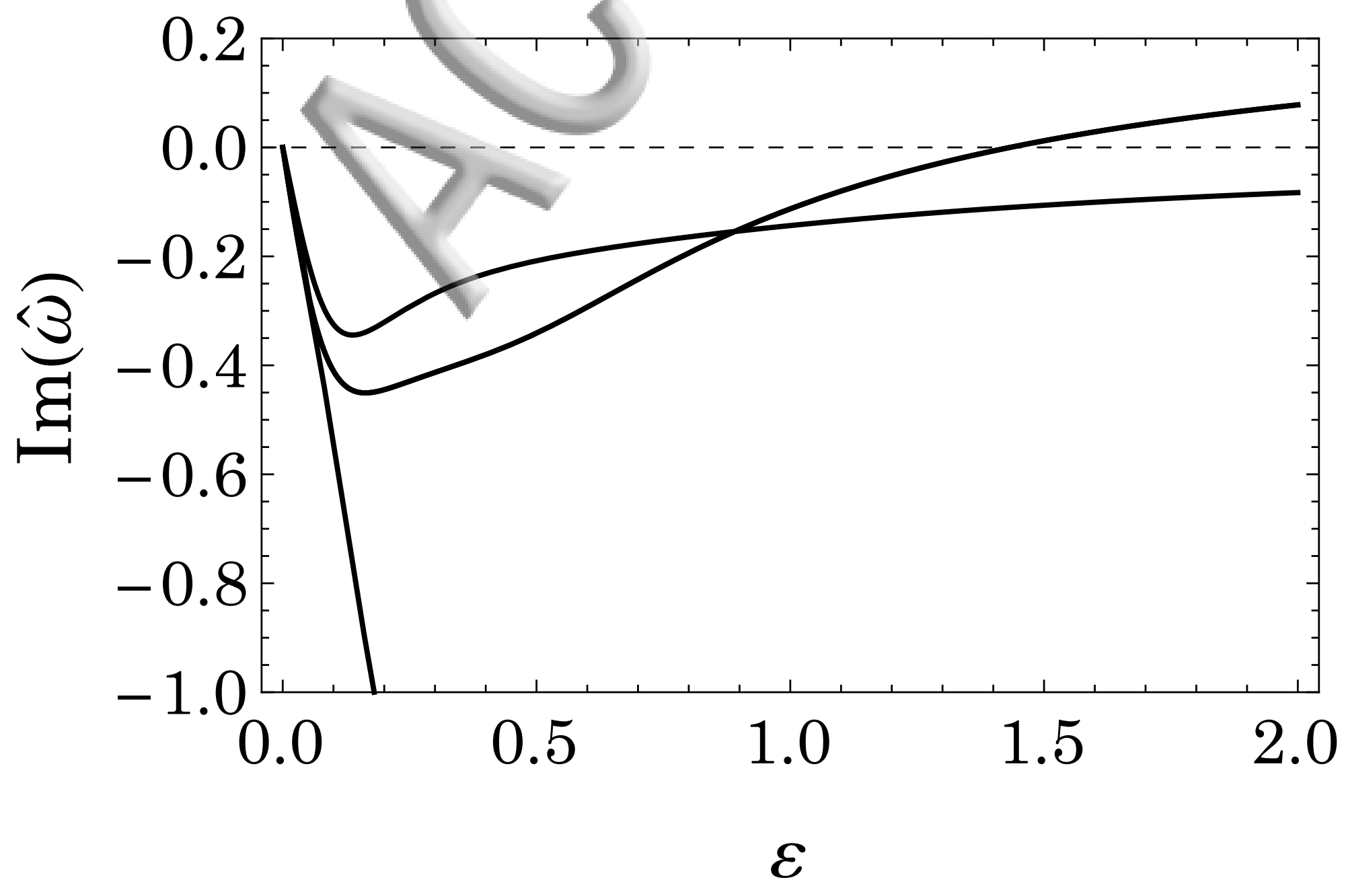
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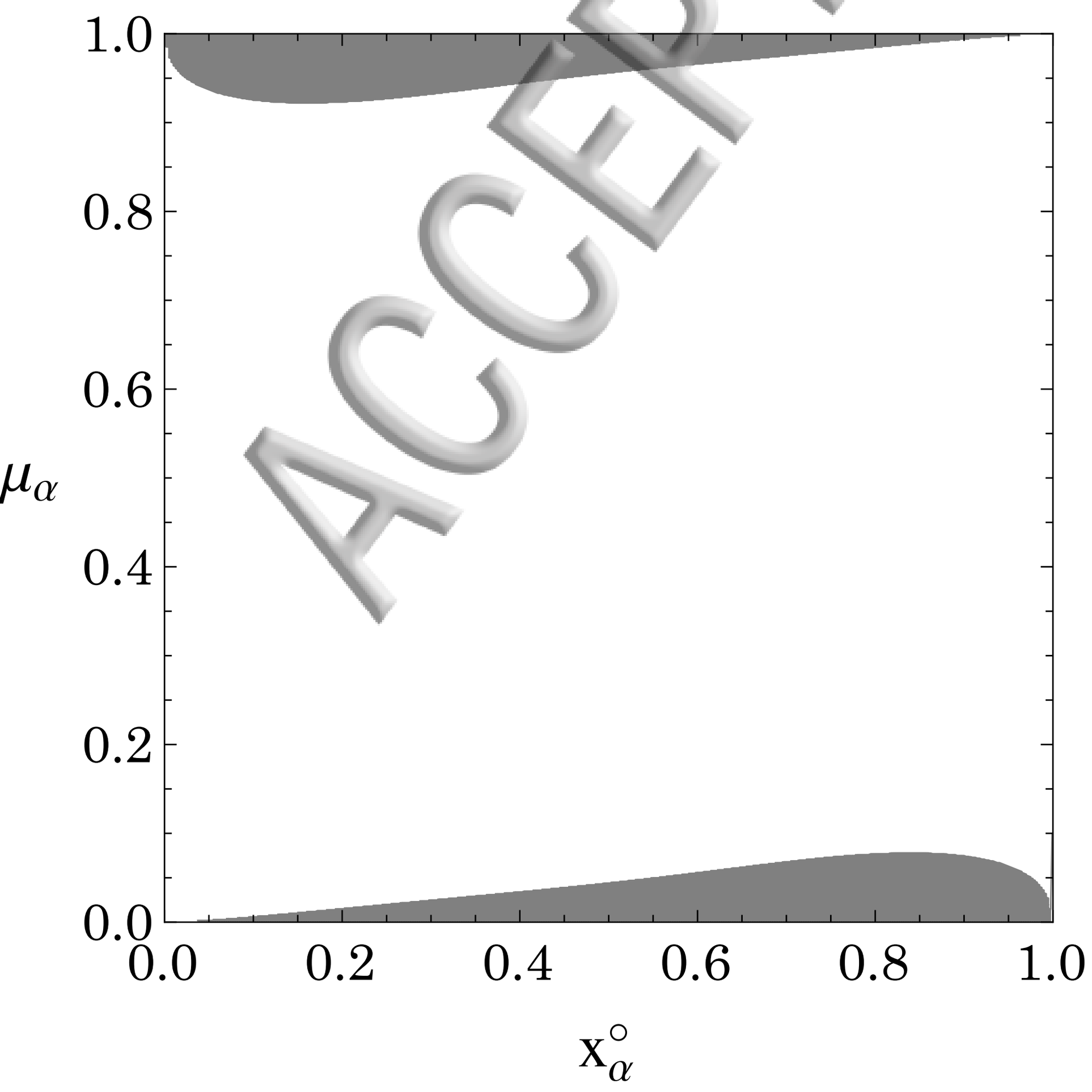
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