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# UNIQUE MOMENT SET FROM THE ORDER OF MAGNITUDE METHOD

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Abstract. The order of magnitude method [Struchtrup, Phys. Fluids 16, 3921-3934 (2004)] is used to construct a unique moment set for 1-D transport with scattering. Simply speaking, the method uses a series of leading order Chapman-Enskog expansions in the Knudsen number to construct the moments such that the number of moments at a given Chapman-Enskog order is minimal. For isotropic scattering, when one begins with monomials for the moments, the method constructs step by step moments of the Legendre polynomials. For anisotropic scattering, however, it constructs moments of new polynomials relevant for the particular scattering mechanism. All terms in the final moment equations are scaled by powers of the Knudsen number, which gives an easy handle to model reduction.

## 1. Introduction.

Ongoing miniaturization of devices requires accurate and fast simulation tools that account for microscale effects. The best known example is hydrodynamics of gases where the classical Navier-Stokes and Fourier transport laws loose validity when the typical system length $L$ is of the order of the gas mean free path $\lambda$, that is when the Knudsen number $\mathrm{Kn} = \lambda/L$ is not sufficiently small [6][11]. The microscopic description of transport through a kinetic equation, e.g., the Boltzmann equation for gases, is accurate at all Knudsen numbers, but numerically expensive.

Models of extended hydrodynamics aim at adding additional terms and equations to the hydrodynamic equations in order to capture microscale effects, but keeping numerical effort relatively low. There are two main routes towards this, the Chapman-Enskog method [2] and Grad's moment method [3], which come both with their own deficiencies. The first order Chapman-Enskog expansion yields classical macroscopic transport laws, e.g. Navier-Stokes-Fourier hydrodynamics, but the higher order expansions, e.g. the Burnett equations [2, 11], usually yield unstable equations [1][15]. Grad's moment method, on the other hand, is not related to the Knudsen number, and thus it is unclear which moment set needs to be considered for a given process [11].

The order of magnitude method was introduced in [8][9] as a combination of Grad's moment method and the Chapman-Enskog expansion. The method yields

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sets of moment equations at any order of the Knudsen number, where the particular moments are produced from the requirement that the number of variables at any order in the Knudsen number is as small as possible. Thus, the order of magnitude method provides the link between Grad-type moment equations and the Knudsen number. Since the method employs Chapman-Enskog expansions only to identify the Knudsen order of moments, the resulting equations are not subject to the stability problems associated with the Chapman-Enskog expansion. In [8][9] the method was applied to the Boltzmann equation, up to third order, where it produced the Regularized 13 Moment Equations (R13) [7]. The R13 equations provide a rather accurate description of rarefied gas flows for moderate Knudsen numbers, see [13] and the references therein.

The present paper aims at making the proceedings of the order of magnitude method transparent by applying it to a simple kinetic equation for one-dimensional radiative transfer with isotropic and anisotropic scattering. It will be seen that for isotropic scattering the required moments are based on Legendre polynomials ($P_\alpha$-moments), but for anisotropic scattering other polynomials have to be used, which depend on the details of the scattering term.

The goal of this paper is not to describe a physically meaningful system, but rather to give a transparent example for the application of the order of magnitude method. It is hoped that the treatment of a simple kinetic equation makes the method more accessible for other researchers. We also point the interested reader to a recent discussion of the method applied not to moment equations but directly to the kinetic equation [4].

As applied below, the order of magnitude method consists of the following steps [8][9]:

1. Set-up of a Grad-type moment system for arbitrary choice and number of moments.
2. First order Chapman-Enskog expansion to determine leading order of moments. Linear combination of moments to construct new moments such that the number of moments at a given Chapman-Enskog order is minimal. Repeat for the next order of magnitude.
3. Use of the established Chapman-Enskog orders to rescale the equations for the new moments, use of the scaling for model reduction to a given order of accuracy.

Step 1 is presented in Section 2 where also the kinetic model is introduced. Step 2 is presented in Section 3 for isotropic scattering and in Section 4 for anisotropic scattering. Step 3 is presented in Section 5. The paper closes with some final comments.

## 2. Kinetic equation and Grad-type moment equations.

### 2.1. Kinetic equation.

We consider one-dimensional transport processes of identical particles that travel with unit velocity in arbitrary directions, $\mu \in (-1,1)$ denotes the direction cosine. The distribution function $f(x,t,\mu)$ obeys the kinetic equation

$$\frac{\partial f}{\partial t}+\mu\frac{\partial f}{\partial x}=-\frac{1}{\varepsilon\tau}\kappa(\mu)(f-f_0)\ , \tag{1}$$

where $\frac{\kappa(\mu)}{\tau}$ is a direction dependent scattering probability, with a dimensionless function $\kappa(\mu)$ and a constant mean free time $\tau$. We consider dimensional quantities and use the scaling parameter $\varepsilon$ which would be the Knudsen number in a dimensionless formulation. With the use of the scaling parameter non-dimensionalization is not necessary. At the end of the discussion, $\varepsilon$ will be set to unity again.

The zeroth moment is conserved (conservation of particle number), that is

$$\frac{1}{\varepsilon\tau}\int\kappa\left(\mu\right)\left(f-f_0\right)d\mu=0\;; \tag{2}$$

here, and in all subsequent integrals, the integration is over the full domain of $\mu$. We require isotropic equilibrium, which means that the local equilibrium distribution $f_0$ is independent of direction. It follows from the conservation condition (2) as

$$f_0=\frac{\int\kappa\left(\mu\right)fd\mu}{\int\kappa\left(\mu\right)d\mu}\;. \tag{3}$$

Interestingly, the local equilibrium distribution depends on the collision probability $\kappa\left(\mu\right)$ as long as non-equilibrium is maintained. In the final equilibrium state (E) both $f$ and $f_0$ are isotropic and equal. Then their value can be determined from the number density which is the zeroth moment:

$$u_0=\int f_E d\mu=\int f_{0,E}d\mu=f_{0,E}\int d\mu=2f_{0,E} \tag{4}$$

so that

$$f_E=f_{0,E}=\frac{1}{2}u_0\;. \tag{5}$$

We show the existence of the H-theorem for the above kinetic equation. Entropy and entropy flux are given by

$$\eta=-\int f^2d\mu\quad,\quad\phi=-\int\mu f^2d\mu \tag{6}$$

so that the entropy balance is

$$\frac{\partial\eta}{\partial t}+\frac{\partial\phi}{\partial x}=\sigma\;. \tag{7}$$

Here, the entropy generation rate is given as

$$\sigma=\frac{1}{\varepsilon\tau}\int\kappa\left(\mu\right)f\left(f-f_0\right)d\mu\;. \tag{8}$$

With the conservation law and the fact that $f_0$ is independent of direction, we have by subtracting zero

$$\begin{aligned}\sigma&=\frac{1}{\varepsilon\tau}\int\kappa\left(\mu\right)f\left(f-f_0\right)d\mu-\frac{1}{\tau\varepsilon}f_0\int\kappa\left(\mu\right)\left(f-f_0\right)d\mu\\&=\frac{1}{\varepsilon\tau}\int\kappa\left(\mu\right)\left(f-f_0\right)^2d\mu\geq0\;.\end{aligned} \tag{9}$$

Thus, entropy generation is positive as long as $\kappa\left(\mu\right)\geq0$ for all $\mu$.

The final kinetic equation is an integro differential equation for the distribution function

$$\frac{\partial f}{\partial t}+\mu\frac{\partial f}{\partial x}=-\frac{1}{\varepsilon\tau}\kappa\left(\mu\right)\left[f-\frac{\int\kappa\left(\mu\right)fd\mu}{\int\kappa\left(\mu\right)d\mu}\right]\;; \tag{10}$$

this equation has one conservation law, and an entropy.

The further proceedings can be performed for any non-negative function $\kappa\left(\mu\right)$. For sake of simplicity we shall use

$$\kappa\left(\mu\right)=1+\gamma\mu^2 \tag{11}$$

with $\gamma = 0$ for isotropic scattering and $\gamma = 1$ for anisotropic scattering. We remark that we do not aim at picturing a relevant scattering mechanism, but rather at having a simple kinetic equation for the application of the order of magnitude method.

2.2. **Moment equations for monomials.** We define the even and odd monomial moments of the distribution as

$$\begin{aligned} u_\alpha &= \int_{-1}^{1} \mu^{2\alpha} f d\mu \quad , \quad \alpha = 0, 1, \ldots, N \\ w_\alpha &= \int_{-1}^{1} \mu^{2\alpha-1} f d\mu \quad , \quad \alpha = 1, \ldots, N \end{aligned} \tag{12}$$

so that we have $(2N+1)$ moments altogether. Multiplying the kinetic equation (10, 11) with powers of $\mu$ and integration gives the nested moment equations

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \frac{\partial w_1}{\partial x} &= 0 \ , \\ \frac{\partial u_\alpha}{\partial t} + \frac{\partial w_{\alpha+1}}{\partial x} &= -\frac{1}{\varepsilon\tau} \left[ u_\alpha + \gamma u_{\alpha+1} - \phi_\alpha \left( u_0 + \gamma u_1 \right) \right] \quad , \quad \alpha = 1, \ldots, N \\ \frac{\partial w_\alpha}{\partial t} + \frac{\partial u_\alpha}{\partial x} &= -\frac{1}{\varepsilon\tau} \left[ w_\alpha + \gamma w_{\alpha+1} \right] \quad , \quad \alpha = 1, \ldots, N \end{aligned} \tag{13}$$

with the coefficients

$$\phi_\alpha = \frac{\frac{1}{2\alpha+1} + \frac{\gamma}{2\alpha+3}}{1 + \frac{\gamma}{3}} \quad , \quad \alpha = 0, 1, \ldots, N \ . \tag{14}$$

The first equation (13) is the conservation law for particle number density $u_0$ with the particle flux $w_1$. Higher moments, and the corresponding equations, do not have a straightforward physical explanation.

As discussed further above, the equilibrium distribution is isotropic, so that the equilibrium moments can be calculated as

$$\begin{aligned} u_{\alpha,E} &= \int \mu^{2\alpha} f_E d\mu = f_E \int \mu^{2\alpha} d\mu = f_E \frac{2}{2\alpha+1} = \frac{u_0}{2\alpha+1} \ , \\ w_{\alpha,E} &= \int \mu^{2\alpha-1} f_E d\mu = 0 \ . \end{aligned} \tag{15}$$

It is easy to see that with these equilibrium values the right hand sides of the moment equations (13) vanish.

2.3. **Grad closure for $2N+1$ moments.** The idea of the Grad moment method is to consider a finite set of moments as variables. For finite moment number $N$ the set of moment equations (13) is not a closed set, since the equations for $\alpha = N$ contain the moments $u_{N+1}$ and $w_{N+1}$. Grad-type closure of the equations requires constitutive relations for $u_{N+1}$ and $w_{N+1}$ which express these as functions of the primary variables $u_\alpha, w_\alpha$ $(\alpha \leq N)$.

We approximate the distribution as a polynomial based on those monomials that are used for creating the moments $u_\alpha, w_\alpha$ $(\alpha \leq N)$,

$$f_G = \sum_{\beta=0}^{N} \upsilon_\beta \mu^{2\beta} + \sum_{\beta=1}^{N} \omega_\beta \mu^{2\beta-1} \ . \tag{16}$$

The polynomial structure can be motivated by determining $f_G$ as that distribution function that maximizes the entropy (6)$_1$ under the constraint of given values of the $(2N+1)$ moments $u_\alpha$ and $w_\alpha$. Then, the coefficients $\upsilon_\beta$ and $\omega_\beta$ in (16) are the Lagrange multipliers of the corresponding isoperimetric problem. Inserting $f_G$ into the definition of the monomial moments gives

$$\begin{aligned} u_\alpha &= \sum_{\beta=0}^{N} \upsilon_\beta \int \mu^{2(\alpha+\beta)} d\mu = \sum_{\beta=0}^{N} \frac{2}{2(\alpha+\beta)+1} \upsilon_\beta \ , \\ w_\alpha &= \sum_{\beta=1}^{N} \omega_\beta \int \mu^{2(\alpha+\beta-1)} d\mu = \sum_{\beta=1}^{N} \frac{2}{2(\alpha+\beta)-1} \omega_\beta \ . \end{aligned} \tag{17}$$

The coefficients $\upsilon_\alpha, \omega_\alpha$ are determined from the first $2N+1$ moments. We introduce the matrices

$$\begin{aligned} \mathcal{A}_{\alpha\beta} &= \frac{2}{2(\alpha+\beta)+1} \quad , \quad \alpha,\beta = 0,\ldots,N \\ \mathcal{B}_{\alpha\beta} &= \frac{2}{2(\alpha+\beta)-1} \quad , \quad \alpha,\beta = 1,\ldots,N \end{aligned} \tag{18}$$

so that upon inversion

$$\begin{aligned} \upsilon_\alpha &= \sum_{\beta=0}^{N} \mathcal{A}^{-1}_{\alpha\beta} u_\beta \quad , \quad \alpha = 0,\ldots,N \\ \omega_\alpha &= \sum_{\beta=1}^{N} \mathcal{B}^{-1}_{\alpha\beta} w_\beta \quad , \quad \alpha = 1,\ldots,N \end{aligned} \tag{19}$$

For the required closure, we thus find

$$u_{N+1} = \sum_{\alpha=0}^{N} \xi_\alpha u_\alpha \quad , \quad w_{N+1} = \sum_{\alpha=1}^{N} \zeta_\alpha w_\alpha \tag{20}$$

with the coefficients

$$\begin{aligned} \xi_\alpha &= \sum_{\beta=0}^{N} \frac{2}{2(N+\beta)+3} \mathcal{A}^{-1}_{\beta\alpha} \quad , \quad \alpha = 0,\ldots,N \\ \zeta_\alpha &= \sum_{\beta=1}^{N} \frac{2}{2(N+\beta)+1} \mathcal{B}^{-1}_{\beta\alpha} \quad , \quad \alpha = 1,\ldots,N \ . \end{aligned} \tag{21}$$

2.4. **Closed equations.** The closed moment set for the monomial moments $u_0$, $u_\alpha$, $w_\alpha$, $\alpha = 1, \ldots, N$ can be written in compact form as

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \frac{\partial w_1}{\partial x} &= 0 \\ \frac{\partial u_\alpha}{\partial t} + \sum_{\beta=1}^{N} R_{\alpha\beta} \frac{\partial w_\beta}{\partial x} &= -\frac{1}{\varepsilon\tau} \sum_{\beta=1}^{N} U_{\alpha\beta} u_\beta + \frac{1}{\tau\varepsilon} \left(\phi_\alpha - \gamma\xi_0\delta_{\alpha N}\right) u_0 \; , \quad \alpha = 1, \ldots, N \\ \frac{\partial w_\alpha}{\partial t} + \frac{\partial u_\alpha}{\partial x} &= -\frac{1}{\varepsilon\tau} \sum_{\beta=1}^{N} W_{\alpha\beta} w_\beta \qquad , \quad \alpha = 1, \ldots, N \end{aligned} \tag{22}$$

where $R_{\alpha\beta}$, $U_{\alpha\beta}$ and $W_{\alpha\beta}$ are $N \times N$ matrices which are given in the appendix, see Eqs. (92, 93, 94).

Similar compact notation would be found for other expressions for the interaction term, which then would lead to different entries in the matrices $U_{\alpha\beta}$ and $W_{\alpha\beta}$ while the matrix $R_{\alpha\beta}$ would remain the same.

## 3. Order of magnitude method, isotropic scattering ($\gamma = 0$).

3.1. **Moment equations.** In the case of isotropic scattering, where $\gamma = 0$, the matrices on the right hand sides of the equations reduce to unit matrices, $W_{\alpha\beta} = U_{\alpha\beta} = \delta_{\alpha\beta}$, and $\phi_\alpha = \frac{1}{2\alpha+1}$. Since the application of the order of magnitude method implies matrix inversions, it becomes much simpler for unit matrices. In fact, in this case we can consider the infinite system of moment equations for monomials, which we write as

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \frac{\partial w_1}{\partial x} &= 0 \\ \frac{\partial u_\alpha}{\partial t} + \frac{\partial w_{\alpha+1}}{\partial x} &= -\frac{1}{\varepsilon\tau} \left( u_\alpha - \frac{u_0}{2\alpha+1} \right) \quad , \quad \alpha = 1, 2, \ldots \\ \frac{\partial w_\alpha}{\partial t} + \frac{\partial u_\alpha}{\partial x} &= -\frac{1}{\varepsilon\tau} w_\alpha \qquad , \quad \alpha = 1, 2, \ldots \end{aligned} \tag{23}$$

3.2. **Equilibrium and 1st order variables.** We begin the application of the order of magnitude method by determining the equilibrium values of the moments. These are obtained by considering (23) in the limit $\varepsilon \to 0$ as

$$u_{\alpha|E} = \frac{u_0}{2\alpha+1} \quad , \quad w_{\alpha|E} = 0 \, . \tag{24}$$

We see that all even moments $u_\alpha$ have an equilibrium value, while all odd moments $w_\alpha$ have no equilibrium value.

The idea of the order of magnitude method is to construct new variables by linear combination such that at any order the number of variables is as small as possible. The first step of this is straightforward: Since the equilibrium values of all moments are given by the value of $u_0$ we require that moment at order zero. The first set of non-equilibrium moments $u_\alpha^{(1)}$, $w_\alpha^{(1)}$ is obtained by subtracting the equilibrium values of the higher moments, as

$$\begin{aligned} u_\alpha^{(1)} &= u_\alpha - u_{\alpha|E} = u_\alpha - \frac{u_0}{2\alpha+1} \quad , \quad \alpha = 1, 2, \ldots \\ w_\alpha^{(1)} &= w_\alpha - w_{\alpha|E} = w_\alpha \qquad , \quad \alpha = 1, 2, \ldots \end{aligned} \tag{25}$$

Moment equations for the new moments are obtained by inserting the new moments (25) into the moment equations (23), and eliminating the time derivative of $u_0$ with the conservation law. This gives the equations

$$\begin{aligned}\frac{\partial u_0}{\partial t}+\frac{\partial w_1^{(1)}}{\partial x} &= 0 \\ \frac{\partial u_\alpha^{(1)}}{\partial t}-\frac{1}{2\alpha+1}\frac{\partial w_1^{(1)}}{\partial x}+\frac{\partial w_{\alpha+1}^{(1)}}{\partial x} &= -\frac{1}{\varepsilon\tau}u_\alpha^{(1)} \quad , \quad \alpha=1,2,\dots \\ \frac{\partial w_\alpha^{(1)}}{\partial t}+\frac{1}{2\alpha+1}\frac{\partial u_0}{\partial x}+\frac{\partial u_\alpha^{(1)}}{\partial x} &= -\frac{1}{\varepsilon\tau}w_\alpha^{(1)} \quad , \quad \alpha=1,2,\dots\end{aligned} \tag{26}$$

3.3. **2nd order variables.** After the first round, the variables are now the equilibrium variable $u_0$, and the non-equilibrium variables $u_\alpha^{(1)}$, $w_\alpha^{(1)}$ which are at least of first order in $\varepsilon$ by construction. We expand only the latter in a Chapman-Enskog series in $\varepsilon$,

$$u_\alpha^{(1)}=\varepsilon u_{\alpha,1}^{(1)}+\varepsilon^2 u_{\alpha,2}^{(1)}+\dots \quad , \quad w_\alpha^{(1)}=\varepsilon w_{\alpha,1}^{(1)}+\varepsilon^2 w_{\alpha,2}^{(1)}+\dots$$

Note that, due to construction in the first round, the variables have no zeroth order contribution. We insert the above into (26), and keep only the leading order terms in $\varepsilon$ to find

$$0=u_{\alpha,1}^{(1)} \quad , \quad -\frac{1}{2\alpha+1}\tau\frac{\partial u_0}{\partial x}=w_{\alpha,1}^{(1)} \quad , \quad \alpha=1,2,\dots \tag{27}$$

Thus, the $u_\alpha^{(1)}$ have no first order contribution while all $w_\alpha^{(1)}$ have first order contributions that are determined by the gradient of $u_0$. Accordingly the $w_\alpha^{(1)}$ are linearly dependent, and we can write

$$w_{\alpha,1}^{(1)}=\frac{3}{2\alpha+1}w_{1,1}^{(1)} \quad , \quad \alpha=2,3,\dots \tag{28}$$

Thus, to first order, all moments can be expressed through the number density $u_0$ and its flux $w_1^{(1)}$, which therefore are considered as base variables. All other moments are replaced by the second non-equilibrium variables

$$\begin{aligned}u_\alpha^{(2)} &= u_\alpha^{(1)} \quad , \quad \alpha=1,2,\dots \\ w_\alpha^{(2)} &= w_\alpha^{(1)}-\frac{3}{2\alpha+1}w_1^{(1)} \quad , \quad \alpha=2,3,\dots\end{aligned} \tag{29}$$

The corresponding transport equations result from (26), after insertion and replacement of the time derivatives of $w_1^{(1)}$

$$\begin{aligned}\frac{\partial u_0}{\partial t}+\frac{\partial w_1^{(1)}}{\partial x} &= 0 \\ \frac{\partial w_1^{(1)}}{\partial t}+\frac{1}{3}\frac{\partial u_0}{\partial x}+\frac{\partial u_1^{(2)}}{\partial x} &= -\frac{1}{\varepsilon\tau}w_1^{(1)} \\ \frac{\partial u_\alpha^{(2)}}{\partial t}+\frac{4\alpha}{(2\alpha+3)(2\alpha+1)}\frac{\partial w_1^{(1)}}{\partial x}+\frac{\partial w_{\alpha+1}^{(2)}}{\partial x} &= -\frac{1}{\varepsilon\tau}u_\alpha^{(2)} \quad , \quad \alpha=1,2,\dots \\ \frac{\partial w_\alpha^{(2)}}{\partial t}-\frac{3}{2\alpha+1}\frac{\partial u_1^{(2)}}{\partial x}+\frac{\partial u_\alpha^{(2)}}{\partial x} &= -\frac{1}{\varepsilon\tau}w_\alpha^{(2)} \quad , \quad \alpha=2,3,\dots\end{aligned} \tag{30}$$

By construction $u_\alpha^{(2)}$, $w_\alpha^{(2)}$ are of second order in the Chapman-Enskog sense, while $w_1^{(1)}$ is of first order and $u_0$ is of zeroth order.

For the next round of the method, it is convenient to indicate the Chapman-Enskog order of the already established variables explicitly by writing $w_1^{(1)} = \varepsilon \hat{w}_1^{(1)}$ and $u_0 = \hat{u}_0$:

$$\begin{aligned}
\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0 \\
\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \frac{\partial u_1^{(2)}}{\partial x} &= -\frac{1}{\tau}\left[\hat{w}_1^{(1)} + \frac{\tau}{3}\frac{\partial \hat{u}_0}{\partial x}\right] \\
\frac{\partial u_\alpha^{(2)}}{\partial t} + \frac{4\alpha}{(2\alpha+3)(2\alpha+1)}\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} + \frac{\partial w_{\alpha+1}^{(2)}}{\partial x} &= -\frac{1}{\varepsilon\tau} u_\alpha^{(2)} \quad , \quad \alpha = 1,2,\ldots \\
\frac{\partial w_\alpha^{(2)}}{\partial t} - \frac{3}{2\alpha+1}\frac{\partial u_1^{(2)}}{\partial x} + \frac{\partial u_\alpha^{(2)}}{\partial x} &= -\frac{1}{\varepsilon\tau} w_\alpha^{(2)} \quad , \quad \alpha = 2,3,\ldots
\end{aligned} \tag{31}$$

3.4. **3rd order variables.** We have now established a variable set with only one variable ($u_0$) at zeroth and one variable ($w_1^{(1)}$) at first order in $\varepsilon$. All other variables $u_\alpha^{(2)}$, $w_\alpha^{(2)}$ are at least of second order. We expand only the second order variables,

$$u_\alpha^{(2)} = \varepsilon^2 u_{\alpha,2}^{(2)} + \varepsilon^3 u_{\alpha,3}^{(2)} + \ldots \quad , \quad w_\alpha^{(2)} = \varepsilon^2 w_{\alpha,3}^{(2)} + \varepsilon^3 w_{\alpha,3}^{(2)} + \ldots \tag{32}$$

and find their leading order contributions by inserting the expansion into (31) as

$$\begin{aligned}
-\frac{4\alpha}{(2\alpha+3)(2\alpha+1)}\tau \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= u_{\alpha,2}^{(2)} \quad , \quad \alpha = 1,2,\ldots \\
0 &= w_{\alpha,2}^{(2)} \quad , \quad \alpha = 2,3,\ldots
\end{aligned} \tag{33}$$

This implies that the $u_{\alpha,2}^{(2)}$ are linear dependent, as

$$u_{\alpha,2}^{(2)} = \frac{15\alpha}{(2\alpha+3)(2\alpha+1)} u_{1,2}^{(2)} \quad , \quad \alpha = 1,2,\ldots \tag{34}$$

while the $w_\alpha^{(2)}$ have no second order contributions. Following the already established pattern, we now introduce the third non-equilibrium variables as

$$\begin{aligned}
u_\alpha^{(3)} &= u_\alpha^{(2)} - \frac{15\alpha}{(2\alpha+3)(2\alpha+1)} u_1^{(2)} \quad , \quad \alpha = 2,3,\ldots \\
w_\alpha^{(3)} &= w_\alpha^{(2)} \quad , \quad \alpha = 2,3,\ldots
\end{aligned} \tag{35}$$

With this round, we now we have the variables $u_0 = \hat{u}_0$, $w_1^{(1)} = \varepsilon \hat{w}_1^{(1)}$, $u_1^{(2)} = \varepsilon^2 \hat{u}_1^{(2)}$ and $u_\alpha^{(3)}, w_\alpha^{(3)}$ with the equations

$$\begin{aligned}
\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0 \\
\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial x} &= -\frac{1}{\tau}\left[\hat{w}_1^{(1)} + \frac{\tau}{3}\frac{\partial \hat{u}_0}{\partial x}\right] \\
\varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial t} + \frac{\partial w_2^{(3)}}{\partial x} &= -\frac{1}{\tau}\varepsilon\left[\hat{u}_1^{(2)} + \frac{4}{15}\tau\frac{\partial \hat{w}_1^{(1)}}{\partial x}\right] \\
\frac{\partial u_\alpha^{(3)}}{\partial t} - \frac{15\alpha}{(2\alpha+3)(2\alpha+1)}\frac{\partial w_2^{(3)}}{\partial x} + \frac{\partial w_{\alpha+1}^{(3)}}{\partial x} &= -\frac{1}{\varepsilon\tau}u_\alpha^{(3)} \quad , \quad \alpha = 2,3,\ldots \\
\frac{\partial w_\alpha^{(3)}}{\partial t} + \frac{9(\alpha-1)}{(2\alpha+3)(2\alpha+1)}\varepsilon^2\frac{\partial \hat{u}_1^{(2)}}{\partial x} + \frac{\partial u_\alpha^{(3)}}{\partial x} &= -\frac{1}{\varepsilon\tau}w_\alpha^{(3)} \quad , \quad \alpha = 2,3,\ldots
\end{aligned} \tag{36}$$

3.5. **4th order variables.** By construction $u_\alpha^{(3)}$, $w_\alpha^{(3)}$ are third order variables, while $u_1^{(2)} = \varepsilon^2 \hat{u}_1^{(2)}$ is second order, $w_0^{(1)} = \varepsilon \hat{w}_0^{(1)}$ is first order and $u_0$ is zeroth order. We expand only the highest order variables,

$$u_\alpha^{(3)} = \varepsilon^3 u_{\alpha,3}^{(3)} + \varepsilon^4 u_{\alpha,4}^{(3)} + \ldots \quad , \quad w_\alpha^{(3)} = \varepsilon^3 w_{\alpha,3}^{(3)} + \varepsilon^4 w_{\alpha,4}^{(3)} + \ldots \tag{37}$$

Again we are interested only in the leading order

$$\begin{aligned}
0 &= u_{\alpha,3}^{(3)} \quad , \quad \alpha = 2,3,\ldots \\
-\frac{9(\alpha-1)}{(2\alpha+3)(2\alpha+1)}\tau\frac{\partial \hat{u}_1^{(2)}}{\partial x} &= w_{\alpha,3}^{(3)} \quad , \quad \alpha = 2,3,\ldots
\end{aligned} \tag{38}$$

The last equation yields

$$w_{\alpha,3}^{(3)} = \frac{35(\alpha-1)}{(2\alpha+3)(2\alpha+1)}w_{2,3}^{(3)} \quad , \quad \alpha = 3,4,\ldots \tag{39}$$

and therefore we introduce the fourth non-equilibrium moments as

$$\begin{aligned}
u_\alpha^{(4)} &= u_\alpha^{(3)} && , \quad \alpha = 2,3,\ldots \\
w_\alpha^{(4)} &= w_\alpha^{(3)} - \frac{35(\alpha-1)}{(2\alpha+3)(2\alpha+1)}w_2^{(3)} && , \quad \alpha = 3,\ldots,N
\end{aligned} \tag{40}$$

$w_2^{(3)} = \varepsilon^3 \hat{w}_2^{(3)}$ is kept as variable. Then we have the moment equations

$$\begin{aligned}
\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0 \\
\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial x} &= -\frac{1}{\tau}\left[\hat{w}_1^{(1)} + \frac{\tau}{3}\frac{\partial \hat{u}_0}{\partial x}\right] \\
\varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial t} + \varepsilon^3 \frac{\partial \hat{w}_2^{(3)}}{\partial x} &= -\varepsilon\frac{1}{\tau}\left[\hat{u}_1^{(2)} + \frac{4}{15}\tau\frac{\partial \hat{w}_1^{(1)}}{\partial x}\right] \\
\varepsilon^3 \frac{\partial \hat{w}_2^{(3)}}{\partial t} + \frac{\partial u_2^{(4)}}{\partial x} &= -\varepsilon^2\frac{1}{\tau}\left[\hat{w}_2^{(3)} + \frac{9}{35}\tau\frac{\partial \hat{u}_1^{(2)}}{\partial x}\right]
\end{aligned} \tag{41}$$

$$\frac{\partial u_\alpha^{(4)}}{\partial t}+\varepsilon^3\frac{40\,(\alpha-1)\,\alpha}{(2\alpha+5)\,(2\alpha+3)\,(2\alpha+1)}\frac{\partial \hat{w}_2^{(3)}}{\partial x}+\frac{\partial w_{\alpha+1}^{(4)}}{\partial x} = -\frac{1}{\varepsilon\tau}u_\alpha^{(4)}\ ,\ \alpha=2,3,\ldots$$
$$\frac{\partial w_\alpha^{(4)}}{\partial t}-\frac{35\,(\alpha-1)}{(2\alpha+3)\,(2\alpha+1)}\frac{\partial u_2^{(4)}}{\partial x}+\frac{\partial u_\alpha^{(4)}}{\partial x} = -\frac{1}{\varepsilon\tau}w_\alpha^{(4)}\ ,\ \alpha=3,4,\ldots$$

3.6. **Summary: Legendre polynomials.** After four rounds of re-assigning variables, we have found the variables

$$u_0=\hat{u}_0\ ,\ \ w_1^{(1)}=\varepsilon\hat{w}_1^{(1)}\ ,\ \ u_1^{(2)}=\varepsilon^2\hat{u}_1^{(2)}\ ,\ \ w_2^{(3)}=\varepsilon^3 w_2^{(3)} \tag{42}$$

as the only variables at Chapman-Enskog orders below $\varepsilon^4$. In the next round the variables $u_\alpha^{(4)}$ and $w_\alpha^{(4)}$ would be expanded to identify the relevant fourth order variables (which would be $u_2^{(4)}$ only) and to construct the 5th order variables $u_\alpha^{(5)}$, $w_\alpha^{(5)}$. Then the next round, and so on. We are confident that the reader now has a clear picture of the procedure, so that further rounds must not be presented in detail.

The variables found from this recurring procedure of leading order Chapman-Enskog expansion and linear combination of moments yields the following variables up to fifth order:

$$\begin{aligned}
u_0 &= \int f d\mu \\
w_1^{(1)} &= \int \mu f d\mu \\
u_1^{(2)} &= u_1-\frac{u_0}{3}=\frac{2}{3}\int\frac{1}{2}\left(3\mu^2-1\right)fd\mu \\
w_2^{(3)} &= w_2-\frac{3}{5}w_1=\frac{2}{5}\int\frac{1}{2}\left(5\mu^3-3\mu\right)fd\mu \\
u_2^{(4)} &= u_2-\frac{6}{7}u_1+\frac{3}{35}u_0=\frac{8}{35}\int\frac{1}{8}\left(35\mu^4-30\mu^2+3\right)fd\mu \\
w_3^{(5)} &= w_3-\frac{70}{63}w_2+\frac{15}{63}w_1=\frac{8}{63}\int\frac{1}{8}\left(63\mu^5-70\mu^3+15\mu\right)fd\mu
\end{aligned} \tag{43}$$

Apart from numerical factors $(1,1,\frac{2}{3},\frac{2}{5},\frac{8}{35},\ldots)$ these are just the moments of the first six Legendre polynomials $P_\alpha$. It is straightforward to conclude that the order of magnitude method constructs the Legendre polynomials as the best set of moments for the kinetic equation (1) with isotropic scattering, in the sense that the number of variables at any given order is as small as possible. We have considered the same kinetic equation before in Ref. [12], where we used $P_\alpha$-moments from the start. Application of the order of magnitude method showed that, indeed, the $P_\alpha$-moment is of order $\alpha$.

The next step in the order of magnitude method is the reduction of the transport equations (41) for the desired level of accuracy. This step is deferred to Section 5, since we will first construct the appropriate variables and equations for the case of anisotropic scattering.

4. **Order of magnitude method, anisotropic scattering ($\gamma\neq 0$).** The application of the order of magnitude method to the case of isotropic scattering as presented above works successively in such a manner, that, in fact, the Grad closure is not needed. This is due to the simple structure of the moment collision

terms, which are independent of the closure ($U_{\alpha\beta}$ and $W_{\alpha\beta}$ are unit matrices). For anisotropic scattering this is not so anymore, since the equations (22) are coupled through the matrices $U_{\alpha\beta}$ and $W_{\alpha\beta}$ in a non-trivial manner. It will be seen that the newly constructed variables, and the corresponding coefficients in the equations, depend on the number of initial moments $N$, and will converge when $N$ is sufficiently large. Again, we will go through four rounds of the procedure, beginning with equilibrium.

4.1. **Equilibrium and first order variables.** The equilibrium values of the moments are obtained by considering the closed set (22) in the the limit $\varepsilon \to 0$. After inversion of the production matrix $U_{\alpha\beta}$ we obtain the equilibrium values as

$$\begin{aligned} u_{\alpha|E} &= \sum_{\beta=1}^{N} U_{\alpha\beta}^{-1} \left(\phi_\beta - \gamma\xi_0\delta_{\beta N}\right) u_0 = \frac{u_0}{2\alpha+1} = \lambda_\alpha^{(1)} u_0 \quad , \quad \alpha = 1,\dots,N \\ w_{\alpha|E} &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \quad \alpha = 1,\dots,N \end{aligned} \tag{44}$$

The above result is checked most easily by inserting the equilibrium values $u_{\alpha|E}$ into the right hand side of $(22)_2$ and use of the definition of matrices and coefficients. As it should be, these are the same equilibrium values as found for the full kinetic equation, see (15). This proves consistency of the Grad closure with the kinetic equation.

All equilibrium moments depend directly on the number density $u_0$ which we chose as the first base variable. As before, we introduce the first non-equilibrium moments as the difference to the local equilibrium values,

$$\begin{aligned} u_\alpha^{(1)} &= u_\alpha - u_{\alpha|E} = u_\alpha - \lambda_\alpha^{(1)} u_0 \quad , \quad \alpha = 1,\dots,N \\ w_\alpha^{(1)} &= w_\alpha - w_{\alpha|E} = w_\alpha \qquad\qquad , \quad \alpha = 1,\dots,N \end{aligned} \tag{45}$$

Replacing the initial moments by the non-equilibrium moments and elimination of the time derivative of $u_0$ by means of the conservation law we obtain the equations

$$\frac{\partial u_0}{\partial t} + \frac{\partial w_1^{(1)}}{\partial x} = 0 \tag{46}$$

$$\frac{\partial u_\alpha^{(1)}}{\partial t} - \lambda_\alpha^{(1)} \frac{\partial w_1^{(1)}}{\partial x} + \sum_{\beta=1}^{N} R_{\alpha\beta} \frac{\partial w_\beta^{(1)}}{\partial x} = -\frac{1}{\varepsilon\tau} \sum_{\beta=1}^{N} U_{\alpha\beta} u_\beta^{(1)} \quad , \quad \alpha = 1,\dots,N$$

$$\frac{\partial w_\alpha^{(1)}}{\partial t} + \lambda_\alpha^{(1)} \frac{\partial u_0}{\partial x} + \frac{\partial u_\alpha^{(1)}}{\partial x} = -\frac{1}{\varepsilon\tau} \sum_{\beta=1}^{N} W_{\alpha\beta} w_\beta^{(1)} \quad , \quad \alpha = 1,\dots,N$$

4.2. **2nd order variables.** After the first round, the variables are now the equilibrium variable $u_0$, and the non-equilibrium variables $u_\alpha^{(1)}$, $w_\alpha^{(1)}$. We expand only the latter in a Chapman-Enskog series in $\varepsilon$,

$$u_\alpha^{(1)} = \varepsilon u_{\alpha,1}^{(1)} + \varepsilon^2 u_{\alpha,2}^{(1)} + \dots \quad , \quad w_\alpha^{(1)} = \varepsilon w_{\alpha,1}^{(1)} + \varepsilon^2 w_{\alpha,2}^{(1)} + \dots \tag{47}$$

We insert the above into (46), and keep only the leading order terms in $\varepsilon$ to find, after matrix inversion,

$$
\begin{aligned}
u_{\beta,1}^{(1)} &= 0 \qquad , \quad \alpha = 1, \ldots, N \\
w_{\alpha,1}^{(1)} &= -\tau \sum_{\beta=1}^{N} W_{\alpha\beta}^{-1} \lambda_{\beta}^{(1)} \frac{\partial u_0}{\partial x} \quad , \quad \alpha = 1, \ldots, N
\end{aligned} \tag{48}
$$

To first order, all odd moments go with the gradient of $u_0$. We can use $w_{1,1}^{(1)}$ to express the others,

$$
w_{\alpha,1}^{(1)} = \frac{\sum_{\beta=1}^{N} W_{\alpha\beta}^{-1} \lambda_{\beta}^{(1)}}{\sum_{\beta=1}^{N} W_{1\beta}^{-1} \lambda_{\beta}^{(1)}} w_{1,1}^{(1)} = \frac{\sum_{\beta=1}^{N} W_{\alpha\beta}^{-1} \lambda_{\beta}^{(1)}}{\kappa^{(1)}} w_{1,1}^{(1)} = \theta_{\alpha}^{(1)} w_{1,1}^{(1)} \tag{49}
$$

where we have defined

$$
\kappa^{(1)} = \sum_{\beta=1}^{N} W_{1\beta}^{-1} \lambda_{\beta}^{(1)} \ . \tag{50}
$$

By construction, the following moments have no first order contribution:

$$
\begin{aligned}
u_{\alpha}^{(2)} &= u_{\alpha}^{(1)} \qquad , \quad \alpha = 1, \ldots, N \\
w_{\alpha}^{(2)} &= w_{\alpha}^{(1)} - \theta_{\alpha}^{(1)} w_{1}^{(1)} = w_{\alpha} - \theta_{\alpha}^{(1)} w_{1}^{(1)} \quad , \quad \alpha = 2, \ldots, N
\end{aligned} \tag{51}
$$

These are introduced into the moment equations, and time derivatives of the flux $w_1^{(1)}$ in higher equations are eliminated by means of its moment equation. To write the resulting equations in a compact form, we introduce the abbreviations

$$
\begin{aligned}
W_{\alpha\beta}^{(2)} &= \left( W_{\alpha\beta} - \theta_{\alpha}^{(1)} W_{1\beta} \right) \quad , \quad \alpha, \beta = 2, \ldots, N \\
\psi_{\alpha}^{(1)} &= \left( \lambda_{\alpha}^{(1)} - \theta_{\alpha}^{(1)} \lambda_{1}^{(1)} \right) \quad , \quad \alpha = 2, \ldots, N \\
\chi_{\alpha}^{(1)} &= \sum_{\beta=1}^{N} R_{\alpha\beta} \theta_{\beta}^{(1)} - \lambda_{\alpha}^{(1)} \quad , \quad \alpha = 1, \ldots, N
\end{aligned} \tag{52}
$$

and make use of the identity

$$
\sum_{\beta=1}^{N} W_{\alpha\beta} \theta_{\beta}^{(1)} = \sum_{\beta=1}^{N} W_{\alpha\beta} \frac{\sum_{\gamma=1}^{N} W_{\beta\gamma}^{-1} \lambda_{\gamma}^{(1)}}{\kappa^{(1)}} = \frac{\lambda_{\alpha}^{(1)}}{\kappa^{(1)}} \ . \tag{53}
$$

After some algebra we find the transport equations

$$
\begin{aligned}
\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0 \\
\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \frac{\partial u_1^{(2)}}{\partial x} &= -\lambda_1^{(1)} \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial u_0}{\partial x} \right] - \frac{1}{\varepsilon\tau} \sum_{\beta=2}^{N} W_{1\beta} w_{\beta}^{(2)}
\end{aligned} \tag{54}
$$

$$
\frac{\partial u_{\alpha}^{(2)}}{\partial t} + \varepsilon \chi_{\alpha}^{(1)} \frac{\partial \hat{w}_1^{(1)}}{\partial x} + \sum_{\beta=2}^{N} R_{\alpha\beta} \frac{\partial w_{\beta}^{(2)}}{\partial x} = -\frac{1}{\varepsilon\tau} \sum_{\beta=1}^{N} U_{\alpha\beta} u_{\beta}^{(2)} \quad , \quad \alpha = 1, \ldots, N
$$

$$\frac{\partial w_\alpha^{(2)}}{\partial t} - \theta_\alpha^{(1)} \frac{\partial u_1^{(2)}}{\partial x} + \frac{\partial u_\alpha^{(2)}}{\partial x} = -\psi_\alpha^{(1)} \left[ \frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] - \frac{1}{\varepsilon\tau} \sum_{\beta=2}^{N} W_{\alpha\beta}^{(2)} w_\beta^{(2)} \ , \qquad \alpha = 2, \ldots, N$$

As before, we have indicated the Chapman-Enskog order of the already established variables explicitly by writing $w_1^{(1)} = \varepsilon \hat{w}_1^{(1)}$ and $u_0 = \hat{u}_0$. All other quantities that appear, that is the $u_\alpha^{(2)}, w_\alpha^{(2)}$, are at least of order $\varepsilon^2$.

4.3. **3rd order variables.** Interestingly, we find the term $\left[ \frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right]$ not only in the equation for $\hat{w}_1^{(1)}$, but also in the equations for the $w_\alpha^{(2)}$. To proceed we have to be careful with the scaling of this expression. A glance at the equation for $\hat{w}_1^{(1)}$ shows that this term vanishes when $\varepsilon$ goes to zero, thus it is at least of first order in $\varepsilon$. To consider this properly in the scaling of the subsequent equations we write

$$\left[ \frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] = \varepsilon \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle \tag{55}$$

where the $\varepsilon$-order is made explicit and the angular-square double bracket indicates an order unity value.

The present variable set has only one variable (number density $u_0$) at zeroth and one variable (flux $w_1^{(1)}$) at first order in $\varepsilon$. All other variables $u_\alpha^{(2)}$, $w_\alpha^{(2)}$ are at least of second order. We expand only the second order moments

$$u_\alpha^{(2)} = \varepsilon^2 u_{\alpha,2}^{(2)} + \varepsilon^3 u_{\alpha,3}^{(2)} + \ldots \quad , \quad w_\alpha^{(2)} = \varepsilon^2 w_{\alpha,2}^{(2)} + \varepsilon^3 w_{\alpha,3}^{(2)} + \ldots \tag{56}$$

and keep leading order to find, after inversion of the matrices $U_{\alpha\beta}$ and $W_{\alpha\beta}^{(2)}$,

$$\begin{aligned} u_{\alpha,2}^{(2)} &= -\left[ \sum_{\beta=1}^{N} U_{\alpha\beta}^{-1} \chi_\beta^{(1)} \right] \tau \frac{\partial \hat{w}_1^{(1)}}{\partial x} && , \quad \alpha = 1, \ldots, N \\ w_{\alpha,2}^{(2)} &= -\left[ \sum_{\beta=2}^{N} W_{\alpha\beta}^{(2)^{-1}} \psi_\beta^{(1)} \right] \tau \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle && , \quad \alpha = 2, \ldots, N \end{aligned} \tag{57}$$

This implies that the $u_{\alpha,2}^{(2)}$ and the $w_{\alpha,2}^{(2)}$ are linear dependent, as

$$\begin{aligned} u_{\alpha,2}^{(2)} &= \frac{\sum_{\beta=1}^{N} U_{\alpha\beta}^{-1} \chi_\beta^{(1)}}{\sum_{\beta=1}^{N} U_{1\beta}^{-1} \chi_\beta^{(1)}} u_{1,2}^{(2)} = \lambda_\alpha^{(2)} u_{1,2}^{(2)} && , \quad \alpha = 1, \ldots, N \\ w_{\alpha,2}^{(2)} &= \frac{\sum_{\beta=2}^{N} W_{\alpha\beta}^{(2)^{-1}} \psi_\beta^{(1)}}{\sum_{\beta=2}^{N} W_{2\beta}^{(2)^{-1}} \psi_\beta^{(1)}} w_{2,2}^{(2)} = \theta_\alpha^{(2)} w_{2,2}^{(2)} && , \quad \alpha = 2, \ldots, N \end{aligned} \tag{58}$$

In the case of anisotropic scattering, the moments $w_\alpha^{(2)}$ have second order contributions, which are absent for the case with isotropic scattering. Following the already established pattern, we now introduce the third non-equilibrium variables as

$$\begin{aligned} u_\alpha^{(3)} &= u_\alpha^{(2)} - \lambda_\alpha^{(2)} u_1^{(2)} \quad , \quad \alpha = 2, \ldots, N \\ w_\alpha^{(3)} &= w_\alpha^{(2)} - \theta_\alpha^{(2)} w_2^{(2)} \quad , \quad \alpha = 3, \ldots, N \end{aligned} \tag{59}$$

As always, we insert the new variables into the transport equations, and eliminate time derivatives in the higher equations. For compact notation we introduce the abbreviations

$$\begin{aligned} R_{\alpha\beta}^{(2)} &= \left( R_{\alpha\beta} - \lambda_\alpha^{(2)} R_{1\beta} \right) \quad , \quad \alpha, \beta = 2, \ldots, N \\ U_{\alpha\beta}^{(2)} &= U_{\alpha\beta} - \lambda_\alpha^{(2)} U_{1\beta} \quad , \quad \alpha, \beta = 2, \ldots, N \\ W_{\alpha\beta}^{(3)} &= W_{\alpha\beta}^{(2)} - \theta_\alpha^{(2)} W_{2\beta}^{(2)} \quad , \quad \alpha, \beta = 3, \ldots, N \\ \kappa^{(2)} &= \sum_{\gamma=2}^{N} W_{2\gamma}^{(2)^{-1}} \psi_\gamma^{(1)} \end{aligned} \tag{60}$$

and we make use of the identities

$$\begin{aligned} \sum_{\beta=2}^{N} W_{\alpha\beta}^{(2)} \theta_\beta^{(2)} &= \frac{\sum_{\beta=2}^{N} W_{\alpha\beta}^{(2)} \sum_{\beta=2}^{N} W_{\beta\gamma}^{(2)^{-1}} \psi_\gamma^{(1)}}{\kappa^{(2)}} = \frac{\psi_\alpha^{(1)}}{\kappa^{(2)}} \\ \sum_{\beta=1}^{N} U_{\alpha\beta} \lambda_\beta^{(2)} &= \frac{\sum_{\beta=1}^{N} U_{\alpha\beta} \sum_{\gamma=1}^{N} U_{\beta\gamma}^{-1} \chi_\gamma^{(1)}}{\mu^{(2)}} = \frac{\chi_\alpha^{(1)}}{\mu^{(2)}} \end{aligned} \tag{61}$$

and

$$\begin{aligned} \sum_{\beta=2}^{N} R_{1\beta} \theta_\beta^{(2)} &= \theta_2^{(2)} = 1 \\ \sum_{\beta=3}^{N} W_{1\beta} w_\beta^{(3)} &= 0 \\ \sum_{\beta=2}^{N} U_{1\beta} u_\beta^{(3)} &= U_{12} u_2^{(3)} = \gamma u_2^{(3)} \\ \sum_{\beta=2}^{N} W_{1\beta} \theta_\beta^{(2)} &= W_{12} \theta_2^{(2)} = \gamma \theta_2^{(2)} = \gamma \\ \lambda_1^{(1)} &= \frac{1}{3} \end{aligned} \tag{62}$$

Making the $\varepsilon$-order of the second order moments explicit by writing $u_1^{(2)} = \varepsilon^2 \hat{u}_1^{(2)}$, $w_2^{(2)} = \varepsilon^2 \hat{w}_2^{(2)}$, we find their transport equations as

$$\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} = 0 \tag{63}$$

$$\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial x} = -\frac{1}{3} \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] - \varepsilon \frac{1}{\tau} \gamma \hat{w}_2^{(2)} \tag{64}$$

$$\varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial t} + \varepsilon^2 \frac{\partial \hat{w}_2^{(2)}}{\partial x} = -\chi_1^{(1)} \varepsilon \left[ \frac{\hat{u}_1^{(2)}}{\tau \mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x} \right] - \frac{1}{\varepsilon \tau} \gamma u_2^{(3)} \tag{65}$$

$$\varepsilon^2 \frac{\partial \hat{w}_2^{(2)}}{\partial t} + \varepsilon^2 \left( \lambda_2^{(2)} - \theta_2^{(1)} \right) \frac{\partial \hat{u}_1^{(2)}}{\partial x} + \frac{\partial u_2^{(3)}}{\partial x} = -\varepsilon \psi_2^{(1)} \left[ \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle + \frac{\hat{w}_2^{(2)}}{\tau \kappa^{(2)}} \right] - \frac{1}{\varepsilon \tau} \sum_{\beta=3}^{N} W_{2\beta}^{(2)} w_\beta^{(3)} \tag{66}$$

The bracketed terms in the third and fourth equation appear also in the higher equations and for the next round we must get their $\varepsilon$-order right. Careful analysis by comparing $\varepsilon$-magnitude of terms in (65, 66) shows that they are in fact of second order, which we make explicit by writing

$$\begin{aligned} \varepsilon \left[ \frac{\hat{u}_1^{(2)}}{\tau \mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x} \right] &= \varepsilon^2 \left\langle \left[ \frac{\hat{u}_1^{(2)}}{\tau \mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x} \right] \right\rangle \\ \varepsilon \left[ \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle + \frac{\hat{w}_2^{(2)}}{\tau \kappa^{(2)}} \right] &= \varepsilon^2 \left\langle \left[ \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle + \frac{\hat{w}_2^{(2)}}{\tau \kappa^{(2)}} \right] \right\rangle \end{aligned} \tag{67}$$

The properly scaled equations for the 3rd order variables $u_\alpha^{(3)}$ and $w_\alpha^{(3)}$ then read

$$\begin{aligned} &\frac{\partial u_\alpha^{(3)}}{\partial t} + \varepsilon^2 \left[ \sum_{\beta=2}^{N} R_{\alpha\beta}^{(2)} \theta_\beta^{(2)} \right] \frac{\partial w_2^{(2)}}{\partial x} + \sum_{\beta=3}^{N} R_{\alpha\beta}^{(2)} \frac{\partial w_\beta^{(3)}}{\partial x} = \\ &\quad = -\varepsilon^2 \left( \chi_\alpha^{(1)} - \lambda_\alpha^{(2)} \chi_1^{(1)} \right) \left\langle \left[ \frac{\hat{u}_1^{(2)}}{\tau \mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x} \right] \right\rangle - \frac{1}{\varepsilon \tau} \sum_{\beta=2}^{N} U_{\alpha\beta}^{(2)} u_\beta^{(3)} \quad , \\ &\qquad \alpha = 2, \ldots, N \end{aligned} \tag{68}$$

$$\begin{aligned} &\frac{\partial w_\alpha^{(3)}}{\partial t} + \varepsilon^2 \left( \lambda_\alpha^{(2)} - \theta_\alpha^{(1)} - \theta_\alpha^{(2)} \left( \lambda_2^{(2)} - \theta_2^{(1)} \right) \right) \frac{\partial \hat{u}_1^{(2)}}{\partial x} - \theta_\alpha^{(2)} \frac{\partial u_2^{(3)}}{\partial x} + \frac{\partial u_\alpha^{(3)}}{\partial x} \\ &= -\varepsilon^2 \left( \psi_\alpha^{(1)} - \theta_\alpha^{(2)} \psi_2^{(1)} \right) \left\langle \left[ \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle + \frac{\hat{w}_2^{(2)}}{\tau \kappa^{(2)}} \right] \right\rangle - \frac{1}{\varepsilon \tau} \sum_{\beta=3}^{N} W_{\alpha\beta}^{(3)} w_\beta^{(3)} \quad , \\ &\qquad \alpha = 3, \ldots, N \end{aligned} \tag{69}$$

The equations (63 - 66, 68, 69) are the counterpart to the set (36) for the case of anisotropic scattering. We see that the equations for anisotropic scattering are more involved, which is due to the more complex form of the scattering matrices $U_{\alpha\beta}$ and $W_{\alpha\beta}$.

4.4. **Approximation for 3rd order variables.** As we proceed with the treatment of the case of anisotropic scattering, the equations become more involved. We shall not go beyond the third order, and of the third order equations we shall only consider the leading terms, which will be determined in the present section.

Again, we perform a Chapman-Enskog expansion of the highest order moments

$$u_\alpha^{(3)} = \varepsilon^3 u_{\alpha,3}^{(3)} + \varepsilon^4 u_{\alpha,4}^{(3)} + \ldots \quad , \quad w_\alpha^{(3)} = \varepsilon^3 w_{\alpha,3}^{(3)} + \varepsilon^4 w_{\alpha,4}^{(3)} + \ldots \tag{70}$$

where we shall determine only the leading terms, i.e., the coefficients $u_{\alpha,3}^{(3)}$, $w_{\alpha,3}^{(3)}$. For a closure of our equations at this level of the proceedings, we will then use only these terms for the variables, that is we shall set

$$u_{\alpha}^{(3)} = \varepsilon^3 \hat{u}_{\alpha}^{(3)} = \varepsilon^3 u_{\alpha,3}^{(3)} \quad , \quad w_{\alpha}^{(3)} = \varepsilon^3 \hat{w}_{\alpha}^{(3)} = \varepsilon^3 w_{\alpha,3}^{(3)} \,. \tag{71}$$

Expansion and inversion of the matrices $U_{\alpha\beta}^{(2)}$ and $W_{\alpha\beta}^{(2)}$ gives $\hat{u}_{\alpha}^{(3)}$ and $\hat{w}_{\alpha}^{(3)}$ in terms of the lower moments $\hat{u}_0$, $\hat{w}_1^{(1)}$, $\hat{u}_1^{(2)}$, $\hat{w}_2^{(2)}$,

$$\hat{u}_{\alpha}^{(3)} = u_{\alpha,3}^{(3)} = -\lambda_{\alpha}^{(3)} \tau \left\langle \left[ \frac{\hat{u}_1^{(2)}}{\tau \mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x} \right] \right\rangle - \bar{\lambda}_{\alpha}^{(3)} \tau \frac{\partial w_2^{(2)}}{\partial x} \quad , \quad \alpha = 2, \ldots, N \tag{72}$$

$$\hat{w}_{\alpha}^{(3)} = w_{\alpha,3}^{(3)} = -\theta_{\alpha}^{(3)} \tau \left\langle \left[ \left\langle \left[ \frac{\hat{w}_1^{(1)}}{\tau \kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x} \right] \right\rangle + \frac{\hat{w}_2^{(2)}}{\tau \kappa^{(2)}} \right] \right\rangle - \bar{\theta}_{\alpha}^{(3)} \tau \frac{\partial \hat{u}_1^{(2)}}{\partial x} \ , \ \alpha = 3, \ldots, N \tag{73}$$

with the new coefficients

$$\begin{aligned}
\lambda_{\alpha}^{(3)} &= \left[ \sum_{\beta=2}^{N} U_{\alpha\beta}^{(2)^{-1}} \left( \chi_{\beta}^{(1)} - \lambda_{\beta}^{(2)} \chi_1^{(1)} \right) \right] && , \quad \alpha = 2, \ldots, N \\
\bar{\lambda}_{\alpha}^{(3)} &= \left[ \sum_{\beta=2}^{N} U_{\alpha\beta}^{(2)^{-1}} \left[ \sum_{\gamma=2}^{N} R_{\beta\gamma}^{(2)} \theta_{\gamma}^{(2)} \right] \right] && , \quad \alpha = 2, \ldots, N \\
\theta_{\alpha}^{(3)} &= \left[ \sum_{\beta=3}^{N} W_{\alpha\beta}^{(3)^{-1}} \left( \psi_{\beta}^{(1)} - \theta_{\beta}^{(2)} \psi_2^{(1)} \right) \right] && , \quad \alpha = 3, \ldots, N \\
\bar{\theta}_{\alpha}^{(3)} &= \left[ \sum_{\beta=3}^{N} W_{\alpha\beta}^{(3)^{-1}} \left( \lambda_{\beta}^{(2)} - \theta_{\beta}^{(1)} - \theta_{\beta}^{(2)} \left( \lambda_2^{(2)} - \theta_2^{(1)} \right) \right) \right] && , \quad \alpha = 3, \ldots, N
\end{aligned} \tag{74}$$

The system of equations for the lower moments $\hat{u}_0$, $\hat{w}_1^{(1)}$, $\hat{u}_1^{(2)}$, $\hat{w}_2^{(2)}$ results from insertion of (72, 73) into (63 - 66). For compact notation, we introduce even more coefficients,

$$\begin{aligned}
\alpha_1 &= \left[ 1 - \gamma \bar{\lambda}_2^{(3)} \right] \\
\alpha_2 &= \left[ \chi_1^{(1)} - \gamma \lambda_2^{(3)} \right] \\
\beta_1 &= \left[ \left( \lambda_2^{(2)} - \theta_2^{(1)} \right) - \sum_{\beta=3}^{N} W_{2\beta}^{(2)} \bar{\theta}_{\beta}^{(3)} - \frac{\lambda_2^{(3)}}{\mu^{(2)}} \right] \\
\beta_2 &= \lambda_2^{(3)} \\
\beta_3 &= \bar{\lambda}_2^{(3)} \\
\beta_4 &= \left[ \psi_2^{(1)} - \sum_{\beta=3}^{N} W_{2\beta}^{(2)} \theta_{\beta}^{(3)} \right] \\
\beta_5 &= \frac{\beta_4}{\kappa^{(2)}}
\end{aligned} \tag{75}$$

With this, we finally obtain a closed system of equations for the first four variables $\hat{u}_0$, $\hat{w}_1^{(1)}$, $\hat{u}_1^{(2)}$, $\hat{w}_2^{(2)}$ that includes the leading order terms of the third order variables:

$$\begin{aligned}
\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0 \\
\varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial x} &= -\frac{1}{3}\left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x}\right] - \frac{1}{\tau}\varepsilon\gamma\hat{w}_2^{(2)} \\
\varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial t} + \varepsilon^2 \alpha_1 \frac{\partial \hat{w}_2^{(2)}}{\partial x} &= -\varepsilon\alpha_2\left[\frac{\hat{u}_1^{(2)}}{\tau\mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x}\right]
\end{aligned} \tag{76}$$

$$\varepsilon^2 \frac{\partial \hat{w}_2^{(2)}}{\partial t} + \varepsilon^2\beta_1 \frac{\partial \hat{u}_1^{(2)}}{\partial x} - \varepsilon^2\beta_2\tau \frac{\partial^2 \hat{w}_1^{(1)}}{\partial x^2} - \varepsilon^3\beta_3\tau\frac{\partial^2 \hat{w}_2^{(2)}}{\partial x^2} = -\varepsilon\beta_4\left\langle\left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x}\right]\right\rangle - \beta_5\varepsilon\frac{\hat{w}_2^{(2)}}{\tau}$$

The use of the given $\varepsilon$-orders for finding the appropriate set of equations for a given order of accuracy will be discussed in Section 5.

4.5. **4th order variables.** From (72, 73) we learn, that the $u_{\alpha,3}^{(3)}$ and the $w_{\alpha,3}^{(3)}$ are linearly dependent. By considering the first two elements of each, we find by yet another inversion

$$\begin{aligned}
u_{\alpha,3}^{(3)} &= \frac{\bar{\lambda}_\alpha^{(3)}\lambda_3^{(3)} - \lambda_\alpha^{(3)}\bar{\lambda}_3^{(3)}}{\bar{\lambda}_2^{(3)}\lambda_3^{(3)} - \lambda_2^{(3)}\bar{\lambda}_3^{(3)}} u_{2,3}^{(3)} + \frac{\lambda_\alpha^{(3)}\bar{\lambda}_2^{(3)} - \bar{\lambda}_\alpha^{(3)}\lambda_2^{(3)}}{\bar{\lambda}_2^{(3)}\lambda_3^{(3)} - \lambda_2^{(3)}\bar{\lambda}_3^{(3)}} u_{3,3}^{(3)} \quad , \quad \alpha = 2,\dots,N \\
w_{\alpha,3}^{(3)} &= \frac{\bar{\theta}_\alpha^{(3)}\theta_4^{(3)} - \theta_\alpha^{(3)}\bar{\theta}_4^{(3)}}{\bar{\theta}_3^{(3)}\theta_4^{(3)} - \theta_3^{(3)}\bar{\theta}_4^{(3)}} w_{3,3}^{(3)} + \frac{\theta_\alpha^{(3)}\bar{\theta}_3^{(3)} - \bar{\theta}_\alpha^{(3)}\theta_3^{(3)}}{\bar{\theta}_3^{(3)}\theta_4^{(3)} - \theta_3^{(3)}\bar{\theta}_4^{(3)}} w_{4,3}^{(3)} \quad , \quad \alpha = 3,\dots,N
\end{aligned} \tag{77}$$

From this we conclude that we have to add $u_2^{(3)}$ and $u_3^{(3)}$, and $w_3^{(3)}$ and $w_4^{(3)}$ to the final variables, and the fourth non-equilibrium variables must be defined as

$$u_\alpha^{(4)} = u_\alpha^{(3)} - \frac{\bar{\lambda}_\alpha^{(3)}\lambda_3^{(3)} - \lambda_\alpha^{(3)}\bar{\lambda}_3^{(3)}}{\bar{\lambda}_2^{(3)}\lambda_3^{(3)} - \lambda_2^{(3)}\bar{\lambda}_3^{(3)}} u_2^{(3)} - \frac{\lambda_\alpha^{(3)}\bar{\lambda}_2^{(3)} - \bar{\lambda}_\alpha^{(3)}\lambda_2^{(3)}}{\bar{\lambda}_2^{(3)}\lambda_3^{(3)} - \lambda_2^{(3)}\bar{\lambda}_3^{(3)}} u_3^{(3)} \quad , \quad \alpha = 4,\dots,N \tag{78}$$

$$w_\alpha^{(4)} = w_\alpha^{(3)} - \frac{\bar{\theta}_\alpha^{(3)}\theta_4^{(3)} - \theta_\alpha^{(3)}\bar{\theta}_4^{(3)}}{\bar{\theta}_3^{(3)}\theta_4^{(3)} - \theta_3^{(3)}\bar{\theta}_4^{(3)}} w_3^{(3)} - \frac{\theta_\alpha^{(3)}\bar{\theta}_3^{(3)} - \bar{\theta}_\alpha^{(3)}\theta_3^{(3)}}{\bar{\theta}_3^{(3)}\theta_4^{(3)} - \theta_3^{(3)}\bar{\theta}_4^{(3)}} w_4^{(3)} \quad , \quad \alpha = 5,\dots,N \tag{79}$$

At this stage, we have the final variables $u_0 = \hat{u}_0$ at zeroth order, $w_1^{(1)} = \varepsilon\hat{w}_1^{(1)}$ at first order, $u_1 = \varepsilon^2\hat{u}_1^{(2)}$ and $w_2^{(2)} = \varepsilon^2\hat{w}_2^{(2)}$ at second order, $u_2^{(3)} = \varepsilon^3\hat{u}_2^{(3)}$, $u_3^{(3)} = \varepsilon^3\hat{u}_3^{(3)}$, $w_3^{(3)} = \varepsilon^3\hat{w}_3^{(3)}$, $w_4^{(3)} = \varepsilon^3\hat{w}_4^{(3)}$ at third order. All other variables, $u_\alpha^{(4)}, w_\alpha^{(4)}$ are at least of fourth order. The transport equations for $u_\alpha^{(4)}$, $w_\alpha^{(4)}$ result from inserting (78, 79) into the moment equations, just as in the previous rounds. We shall not go there, however, but continue with the discussion of the set (76) which incorporates only the leading order of the $u_\alpha^{(3)}$, $w_\alpha^{(3)}$.

| | $\gamma = 0$ | $\gamma = 1$ | | | | $\gamma = 5$ |
|---|---|---|---|---|---|---|
| | | $N = 1$ | $N = 2$ | $N = 4$ | $N = 10$ | $N = 10$ |
| $\kappa^{(1)}$ | $\frac{1}{3} = 0.333$ | 0.2083 | 0.2144 | 0.2146 | 0.2146 | 0.09712 |
| $\alpha_1$ | 1 | $n/a$ | 0.7843 | 0.8263 | 0.8266 | 0.6029 |
| $\alpha_2$ | $\frac{4}{15} = 0.2667$ | $n/a$ | 0.2160 | 0.2135 | 0.2135 | 0.1316 |
| $\mu^{(2)}$ | $\frac{4}{15} = 0.2667$ | 0.1830 | 0.1619 | 0.1606 | 0.1606 | 0.08338 |
| $\beta_1$ | $\frac{9}{35} = 0.2571$ | $n/a$ | 0.1790 | 0.1764 | 0.1766 | 0.1232 |
| $\beta_2$ | 0 | $n/a$ | 0.0053 | 0.0065 | 0.0065 | 0.0043 |
| $\beta_3$ | $\frac{16}{63} = 0.2440$ | 1 | 0.2157 | 0.1737 | 0.1734 | 0.07943 |
| $\beta_4$ | 0 | $n/a$ | 0.0151 | 0.0161 | 0.0163 | 0.03190 |
| $\beta_5$ | 1 | $n/a$ | 1.5565 | 1.5223 | 1.539 | 2.5678 |

TABLE 1. Coefficients for the equations (76) for isotropic scattering ($\gamma = 0$) and for anisotropic scattering ($\gamma = 1, 10$) for various number $N$ of base moments.

4.6. **Numerical values of coefficients.** The coefficients $\alpha_{1,2}$ and $\beta_{1-5}$ in the truncated transport equations (76) depend on the details of the scattering term and the Grad closure through the matrices $R_{\alpha\beta}$, $U_{\alpha\beta}$ and $W_{\alpha\beta}$. For isotropic scattering ($\gamma = 0$) the coefficients are those found before in Section 3. For anisotropic scattering the coefficients depend on the strength of anisotropy $\gamma$ and on the number of moments $N$ used as a base in the Grad method. Table 1 shows the coefficient values for $\gamma = 0$, for $\gamma = 1$ with several values of $N \leq 10$, and for $\gamma = 5, N = 10$. Convergence of coefficients with increasing moment number $N$ is clearly seen, further increase of $N$ does not change the results. The table also shows that the coefficients differ noticeably between isotropic and anisotropic scattering.

5. **Model reduction by order of accuracy.** In the previous sections we have systematically derived a system of transport equations that includes leading terms of the third order moments. While the resulting equations exhibit only the four variables $\hat{u}_0$, $\hat{w}_1^{(1)}$, $\hat{u}_1^{(2)}$, $\hat{w}_2^{(2)}$ they are a condensed form of a full Grad-type moment system with $(2N + 1)$ moments. By application of the order of magnitude method new variables were designed such that at a given order in $\varepsilon$ the number of variables is minimal. Thus, the final equations (76) contain all elements from the $(2N + 1)$ moments that are of third order.

Depending on the process to be described by the equations, it might suffice to consider reduced systems, were reduction is based on the smallness parameter $\varepsilon$. In the following subsections we shall show how this last step of the order of magnitude methods is performed on the scaled equations (76). For this, we define the order of accuracy of a set of equations as follows [8]: A set of equations is said to be accurate of order $\eta$, when the energy flux $w_1 = w_1^{(1)}$ is known within the order $\mathcal{O}(\varepsilon^\eta)$.

5.1. **Zeroth order accuracy.** We have seen that the energy flux is of leading order $\varepsilon$, $w_1 = w_1^{(1)} = \varepsilon \hat{w}_1^{(1)}$. Thus, the flux vanishes to zeroth order in $\varepsilon$. Trivially, the corresponding equation gives the steady state,

$$\frac{\partial u_0}{\partial t} = 0 \,. \tag{80}$$

5.2. **First order accuracy.** Now we consider the first order in the smallness parameter $\varepsilon$. The first equation we need is the conservation law (76)$_1$

$$\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} = 0\,. \tag{81}$$

According to our definition of order of accuracy, we require the flux to first order. This is extracted from the equation (76)$_2$ from which we only require the leading order contribution (which has order $\varepsilon^0$),

$$0 = -\frac{1}{3}\left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x}\right]\,. \tag{82}$$

Combination of these two equations and removal of the $\varepsilon$-scaling (remove hats, set $\varepsilon = 1$) gives the diffusion equation for $u_0$,

$$\frac{\partial u_0}{\partial t} - \tau\kappa^{(1)}\frac{\partial^2 u_0}{\partial x^2} = 0\,. \tag{83}$$

Note that the diffusion coefficient $\tau\kappa^{(1)}$ is computed from the full set of $(2N+1)$ moment equations, see (50). The diffusion equation is the classical transport law associated with the kinetic equation (1).

5.3. **Second order accuracy.** To increase the order of accuracy we have to consider higher order contributions to the flux $\hat{w}_1^{(1)}$. For second order, we need the conservation law and the two leading terms (of order $\varepsilon^0$ and $\varepsilon^1$) of the transport equation for the flux (76)$_2$,

$$\begin{aligned} \frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0\,, \\ \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} &= -\frac{1}{3}\left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x}\right] - \varepsilon\frac{1}{\tau}\gamma\hat{w}_2^{(2)}\,. \end{aligned} \tag{84}$$

Since the second equation contains the term $\varepsilon\hat{w}_2^{(2)}$, the leading order of $\hat{w}_2^{(2)}$ is required as well. The leading terms of the corresponding transport equation (76)$_4$ give

$$\hat{w}_2^{(2)} = -\frac{\beta_4}{\beta_5}\tau\left\langle\left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x}\right]\right\rangle\,. \tag{85}$$

Combining the three equations above, we find a hyperbolic system that reads ($\varepsilon$ scaling removed)

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \frac{\partial w_1^{(1)}}{\partial x} &= 0\,, \\ \frac{\partial w_1^{(1)}}{\partial t} + \left(\frac{1}{3} - \gamma\kappa^{(2)}\right)\frac{\partial u_0}{\partial x} &= -\left(\frac{1}{3} - \gamma\kappa^{(2)}\right)\frac{w_1^{(1)}}{\tau\kappa^{(1)}}\,. \end{aligned} \tag{86}$$

While the above contains only the equilibrium moment $u_0$ and its flux $w_1^{(1)}$, the influence of other moment equations is manifest through the coefficients $\kappa^{(1)}$ and $\kappa^{(2)} = \frac{\beta_4}{\beta_5}$. We emphasize that the Grad moment system for the variables $u_0$ and $w_1^{(1)}$ alone would yield different values of the coefficients as long as anisotropic scattering plays a role. Indeed, for $\gamma = 1$ we find converged values $\kappa^{(1)} = 0.2146$ and $\kappa^{(2)} = 0.0106$ whereas the Grad method with only $u_0$ and $w_1$ would give $\kappa^{(1)} =$

$\frac{5}{15+9\gamma} = 0.2083$ and $\kappa^{(2)} = 0$. For $\gamma = 5$ we find converged values $\kappa^{(1)} = 0.09712$ and $\kappa^{(2)} = 0.01242$ whereas the Grad method with only $u_0$ and $w_1$ would give $\kappa^{(1)} = \frac{5}{15+9\gamma} = 0.08333$ and $\kappa^{(2)} = 0$.

5.4. **Third order accuracy.** For the third order set, we require the conservation law $(76)_1$ and the full equation $(76)_2$ for $w_1^{(1)}$

$$\begin{aligned}\frac{\partial \hat{u}_0}{\partial t} + \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial x} &= 0 \\ \varepsilon \frac{\partial \hat{w}_1^{(1)}}{\partial t} + \varepsilon^2 \frac{\partial \hat{u}_1^{(2)}}{\partial x} &= -\frac{1}{3}\left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial \hat{u}_0}{\partial x}\right] - \varepsilon \frac{1}{\tau}\gamma \hat{w}_2^{(2)}\end{aligned} \tag{87}$$

The second order variable $\hat{u}_1^{(2)}$ appears with the factor $\varepsilon^2$. Thus, at the present order it is sufficient to have its leading term, which is obtained from $(76)_3$ as

$$0 = -\alpha_2 \left[\frac{\hat{u}_1^{(2)}}{\tau\mu^{(2)}} + \frac{\partial \hat{w}_1^{(1)}}{\partial x}\right] . \tag{88}$$

The other second order variable $\hat{w}_2^{(2)}$, however, appears with the factor $\varepsilon^1$. Thus, to have $\varepsilon^2$ accuracy in the equation for $\hat{w}_1^{(1)}$ we need not only the leading term for $\hat{w}_2^{(2)}$ but also the first correction. In other words, the proper equation for $\hat{w}_2^{(2)}$ is given by the two leading terms (of order $\varepsilon^1$ and $\varepsilon^2$) of $(76)_4$,

$$\varepsilon^2 \frac{\partial \hat{w}_2^{(2)}}{\partial t} + \varepsilon^2 \beta_1 \frac{\partial \hat{u}_1^{(2)}}{\partial x} - \varepsilon^2 \beta_2 \tau \frac{\partial^2 \hat{w}_1^{(1)}}{\partial x^2} = -\varepsilon \beta_4 \left\langle \left[\frac{\hat{w}_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial u_0}{\partial x}\right] \right\rangle - \varepsilon \beta_5 \frac{\hat{w}_2^{(2)}}{\tau} \tag{89}$$

The resulting system is not hyperbolic but of a mixed type: some regularizing second order derivatives appear in the equations for $w_1^{(1)}$ and $w_2^{(2)}$ ($\varepsilon$ scaling removed):

$$\begin{aligned}\frac{\partial u_0}{\partial t} + \frac{\partial w_1^{(1)}}{\partial x} &= 0 \\ \frac{\partial w_1^{(1)}}{\partial t} - \tau\mu^{(2)} \frac{\partial^2 w_1^{(1)}}{\partial x^2} &= -\frac{1}{3}\left[\frac{w_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial u_0}{\partial x}\right] - \frac{1}{\tau}\gamma w_2^{(2)} \\ \frac{\partial w_2^{(2)}}{\partial t} - \left(\mu^{(2)}\beta_1 + \beta_2\right)\tau \frac{\partial^2 w_1^{(1)}}{\partial x^2} &= -\beta_4 \left[\frac{w_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial u_0}{\partial x}\right] - \beta_5 \frac{w_2^{(2)}}{\tau}\end{aligned} \tag{90}$$

The last equation is coupled to the first two only through the right hand side of the second equation, $\gamma w_2^{(2)}$. This coupling vanishes for isotropic scattering, where $\gamma = 0$. Then, only the first two equations are required (with $\kappa^{(1)} = \frac{1}{3}$ and $\mu^{(2)} = \frac{4}{15}$).

In fact, we recover here, by rigorous derivation based on the order of magnitude of moments and the order of accuracy of equations, a diffusive correction of higher order (the expressions with second derivatives) which were proposed recently by means of a different argument [14][5]. Interestingly, in [14] it was shown that the diffusive closure gives a significant improvement of the closure.

5.5. **Fourth order accuracy.** Finally, we consider the fourth order of accuracy. For this, we have to add the next higher order terms in the equations for $\hat{u}_1^{(2)}$

and $\hat{w}_2^{(2)}$. This gives the full system (76), which we repeat here with the $\varepsilon$-scaling removed:

$$\begin{aligned}
\frac{\partial u_0}{\partial t} + \frac{\partial w_1^{(1)}}{\partial x} &= 0 \qquad (91)\\
\frac{\partial w_1^{(1)}}{\partial t} + \frac{\partial u_1^{(2)}}{\partial x} &= -\frac{1}{3}\left[\frac{w_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial u_0}{\partial x}\right] - \frac{1}{\tau}\gamma w_2^{(2)}\\
\frac{\partial u_1^{(2)}}{\partial t} + \alpha_1 \frac{\partial w_2^{(2)}}{\partial x} &= -\alpha_2\left[\frac{u_1^{(2)}}{\tau\mu^{(2)}} + \frac{\partial w_1^{(1)}}{\partial x}\right]\\
\frac{\partial w_2^{(2)}}{\partial t} + \beta_1 \frac{\partial u_1^{(2)}}{\partial x} - \beta_2\tau\frac{\partial^2 \hat{w}_1^{(1)}}{\partial x^2} - \beta_3\tau\frac{\partial^2 w_2^{(2)}}{\partial x^2} &= -\beta_4\left[\frac{w_1^{(1)}}{\tau\kappa^{(1)}} + \frac{\partial u_0}{\partial x}\right] - \beta_5\frac{w_2^{(2)}}{\tau}
\end{aligned}$$

Again, this system has a diffusive closure [14][5], now at higher order. Recall that $u_0$ is the original equilibrium variable (number density) and $w_1^{(1)} = w_1$ is its flux. The other variables $u_1^{(2)}$ , $w_2^{(2)}$ are higher order moments that describe non-local effects. The equations were designed such that all information of the kinetic equation up to 4th order accuracy is included. For this all moments were needed up to their third order terms. Should one be interested in higher order equations, one first has to go back to Section 4 and add another round of reconstruction of variables, and so on.

6. **Discussion.** We have applied the order of magnitude method for the construction of macroscopic transport equations at a given order in the Knudsen number to a kinetic equation with scattering. Due to the simplicity of the kinetic description, the three steps of the method are relatively easy to perform. These are: Step 1: Construction of a Grad-type system of moment equations of arbitrary size. Step 2: Construction of moments such that the number of variables at each order in the Knudsen number (in the Chapman-Enskog sense) is as small as possible. Step 3: Reduction of the scaled equations for the new variables to the required order of accuracy in the Knudsen number.

The equations at any order are based on a larger (or, for isotropic scattering, even infinite) set of moment equations. The order of magnitude method condenses the larger system into the reduced one by retaining all elements of all moment equations of the initial system that are needed for the desired level of accuracy.

An important feature of the method it that the relevant variables at any order are constructed step by step. No intuition is required of what moments one should use. For our simple kinetic model we used monomial moments to construct the initial moment system, and found that the proper moments to use for the case of isotropic scattering are $P_\alpha$ moments (Legendre polynomials). However, the constructed variables depend on the scattering process, and for anisotropic scattering a different moment set (not $P_\alpha$ moments) is produced.

The resulting equations resemble moment equations of Grad type, only that the specially constructed variables appear. The final equations at a given order of accuracy are partial differential equations in time and space, where time derivatives appear only in first order, and space derivatives in first and second order. In contrast to this, the application of the Chapman-Enskog expansion to higher orders would yield equations with higher and higher order space derivatives [12], which are problematic numerically, and which are associated with the stability problems of the Chapman-Enskog method. These problems are avoided here, and it can be expected that the equations are stable at any order [8][11].

Another, maybe even more important, aspect of the equations obtained from the order of magnitude method is that they give a proper description of Knudsen boundary layers. Indeed, Knudsen layers which describe mean free path effects close to walls and interfaces are not included in the Chapman-Enskog scaling, which considers the gas only in the bulk. Therefore, equations derived by the Chapman-Enskog method typically fail in the description of Knudsen layers [10]. The equations that result from the order of magnitude method, however, have a moment structure, and they retain the ability of sufficiently large sets of moment equations to describe Knudsen layers. We refer the reader to Ref. [12], where we have considered the same moment equations as obtained here for isotropic scattering, that is the $P_\alpha$- equations. It was shown that, when furnished with meaningful boundary conditions, the equations exhibit Knudsen layers, where already a small number of moments gives a remarkably good description, and a marked improvement over theories that do not describe Knudsen layers.

The order of magnitude method was applied earlier to the Boltzmann equation to find the R13 equations for Maxwell molecules [8], and the proper form of Grad's 13 moment equations for arbitrary particle interaction potentials [9]. The method can be applied to any kinetic equation for processes for which the Knudsen number is finite. We hope that the present application to a simple kinetic equation will serve as a teaching tool for researchers interested in the method, which should find more interesting applications, e.g., to gas mixtures or polyatomic gases, in the future.

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**Appendix** A. **Matrices in the Grad closure.** Here we present the matrices $R_{\alpha\beta}$, $U_{\alpha\beta}$ and $W_{\alpha\beta}$ for the Grad closure. The matrix $R_{\alpha\beta}$ describes the flux on the left hand side, and is not dependent on the scattering probability (all non-specified matrix elements are zero):

$$R_{\alpha\beta} = \begin{bmatrix} 0 & 1 & & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & 1 & & & & \\ & & & \ddots & \ddots & & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 0 & 1 \\ \zeta_1 & \zeta_2 & \zeta_3 & \cdots & & \cdots & \zeta_{N-1} & \zeta_N \end{bmatrix}_{N\times N} \tag{92}$$

The matrices $U_{\alpha\beta}$ and $W_{\alpha\beta}$ that describe the change of moments due to scattering depend on the expression for the scattering term. With $\kappa(\mu) = 1 + \gamma\mu^2$ they read (all non-specified matrix elements are zero):

$$U_{\alpha\beta} = \begin{bmatrix} 1-\gamma\phi_1 & \gamma & & & & & & \\ -\gamma\phi_2 & 1 & \gamma & & & & & \\ -\gamma\phi_3 & & 1 & \gamma & & & & \\ & & & \ddots & \ddots & & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & \gamma \\ \gamma\xi_1-\gamma\phi_N & \gamma\xi_2 & \gamma\xi_3 & \cdots & & & \gamma\xi_{N-1} & 1+\gamma\xi_N \end{bmatrix}_{N\times N} \tag{93}$$

$$W_{\alpha\beta} = \begin{bmatrix} 1 & \gamma & & & & & & \\ & 1 & \gamma & & & & & \\ & & 1 & \gamma & & & & \\ & & & \ddots & \ddots & & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & \gamma \\ \gamma\zeta_1 & \gamma\zeta_2 & \gamma\zeta_3 & \cdots & & \cdots & \gamma\zeta_{N-1} & 1+\gamma\zeta_N \end{bmatrix}_{N\times N} \tag{94}$$

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