Comment on "Thermodynamically Admissible 13 Moment Equations from the Boltzmann Equation"

Rarefied gas flows cannot be described by the equations of classical hydrodynamics—the laws of Navier-Stokes and Fourier—and require extended transport models, which can be derived from the Boltzmann equation.

In a recent Letter, Öttinger [1] presents a nonlinear set of 13 moment equations with H theorem for the description of rarefied gas flows. His derivation is based on the existence of an entropy conserved by reversible processes and the possibility of a Hamiltonian formulation of reversible dynamics. While Öttinger focuses on the thermodynamic structure of the equations, he relaxes the relation to the Boltzmann equation and does not ask whether his equations produce physically meaningful results. We show below that his equations fail to describe a basic transport mechanism: heat transfer at steady state.

Possibly, an extension of Öttinger's approach could lead to a nonlinear entropy for the regularized 13 moment (R13) equations (see [2–5] and references therein), which have been shown to describe all transport mechanisms with good accuracy, including steady state heat transfer and all known rarefaction effects, and which reduce to the equations of hydrodynamics in the limit of small Knudsen numbers. Presently, the R13 equations have an accompanying H theorem only for the linear case [3]; an H theorem for the nonlinear case would further increase the confidence in the moment approach.

While Öttinger's equations possess an H theorem for the nonlinear case, they do not include classical hydrodynamics in the limit of small Knudsen numbers. The hydrodynamic limit of his equations is given in [1] as (with $\mathrm{tr}\pi = 3mk_BT$ and a correction for the trace-free velocity gradient)

$$\pi = mk_BT\mathbf{1} - mk_BT\tau\left(\kappa + \kappa^T - \frac{2}{3}\,\mathrm{tr}\kappa\mathbf{1}\right), \qquad (1)$$

$$\varphi = -\frac{\tau}{nk_BTb\bar{D}}\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{j}^q. \qquad (2)$$

The first equation is the proper Navier-Stokes law, and the second equation should describe heat transfer, but does not. With the definition $\mathbf{q} = \mathbf{Q}{:}\pi$ [1], the coefficients in Eq. (15) of Ref. [1] are related as $\bar{Q}_2 = (5\bar{Q}_1 - 1)/\varphi$. Then, Eqs. (15)–(17) give to leading order

$$\mathbf{j}^q = \frac{\rho k_BT}{2m^2}\mathbf{q} \quad \text{and} \quad \varphi = \mathbf{q}\cdot\pi^{-1}\cdot\mathbf{q} = \frac{4(\mathbf{j}^q)^2}{\rho^2(\frac{k_BT}{m})^3}. \qquad (3)$$

By combining the above equations, one finds

$$(\mathbf{j}^q)^2 = -\frac{\tau}{b\bar{D}}\frac{\rho}{4}\left(\frac{k_B}{m}T\right)^2\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{j}^q. \qquad (4)$$

This equation does not imply Fourier's law, $\mathbf{j}^q = -\lambda(T)\frac{\partial T}{\partial\mathbf{r}}$ [2], which, in fact, cannot be extracted from the equations. For steady state heat transfer in a gas at rest, the energy balance—the trace of Eq. (14) (Ref. [1])—reduces to $\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{j}^q = 0$, and Eq. (4) implies $\mathbf{j}^q = 0$, that is no heat transfer at all. Also, the fully nonlinear equations fail to describe heat transfer, but we have no room for details.

Moreover, the entropy inequality (19) of Ref. [1] lacks a term with nonconvective entropy flux. It is clear from the Boltzmann equation that such a nonconvective entropy flux $\mathbf{j}^s$ is expected; in the hydrodynamic limit it should reduce to $\mathbf{j}^s = \mathbf{j}^q/T$ [2]. Likely, the failure to describe heat transfer is linked to the missing entropy flux.

All problems mentioned originate from Eq. (18) of Ref. [1], which is missing crucial terms. Indeed, the equation differs from the corresponding one that is obtained from the moment equations of the Boltzmann equation; the latter should serve as a guideline to correct Eq. (18).

The problematic structure of (18) is further reflected in the division by φ: Since $\varphi = \mathbf{q}\cdot\pi^{-1}\cdot\mathbf{q}$, and $\mathbf{q}$ vanishes in equilibrium, in close-to-equilibrium processes the term in the second line of the equation becomes dominant over all other terms. This gives another hint that heat transfer is not properly described. It also complicates the linearization of the equations and implies singular signal speeds close to equilibrium, which poses mathematical and practical problems to the solution of the equation.

We believe that the basic ideas of [1], in particular, the variable transforms by means of the tensor π, will open the door to finding H theorems for moment equations. The presented 13 moment equations with entropy, however, suffer from the following problems: They cannot describe heat transfer, they cannot be reduced to the equations of classical hydrodynamics, their structure is not in accordance with the Boltzmann equation, they cannot be properly linearized, and their entropy inequality does not include a nonconvective entropy flux.

Henning Struchtrup[1] and Manuel Torrilhon[2]
[1]Department of Mechanical Engineering
University of Victoria
Victoria BC, Canada
[2]Center for Computational Engineering Sciences
RWTH Aachen
Aachen, Germany

Received 8 July 2010; published 17 September 2010
DOI: 10.1103/PhysRevLett.105.128901
PACS numbers: 05.70.Ln, 47.10.−g, 47.45.Ab, 51.10.+y

[1] H. C. Öttinger, Phys. Rev. Lett. **104**, 120601 (2010).
[2] H. Struchtrup, *Macroscopic Transport Equations for Rarefied Gas Flows* (Springer, Heidelberg, 2005).
[3] H. Struchtrup and M. Torrilhon, Phys. Rev. Lett. **99**, 014502 (2007).
[4] M. Torrilhon and H. Struchtrup, J. Comput. Phys. **227** 1982 (2008).
[5] P. Taheri, M. Torrilhon, and H. Struchtrup, Phys. Fluids **21**, 017102 (2009).

