Higher Order Bulk and Boundary Effects in Channel Flows

Henning Struchtrup

Dept. of Mechanical Engineering, University of Victoria, PO Box 3055 STN CSC, Victoria BC V8W 3P6, Canada, e-mail: struchtr@uvic.ca

Abstract. The regularized 13 moment (R13) equations and their boundary conditions are considered for plane channel flows. Chapman-Enskog scaling based on the Knudsen number is used to reduce the equations. The reduced equations yield second order slip conditions, and allow to describe the characteristic dip in the temperature profile observed in force driven Poiseuille flow.

Keywords: Rarefied gas flows, Microchannels, Slip flow, Regularized 13-moment equations
PACS: 51.10.+y, 47.10.ab, 47.45.Gx, 05.70.Ln

INTRODUCTION

A classical question in rarefied gas dynamics is whether macroscopic transport equations can serve to describe rarefied gas flows. The derivation of the Navier-Stokes and Fourier laws by means of the first order Chapman-Enskog expansion was one of the early successes of kinetic theory. However, the extension to higher orders, which leads to the Burnett and super-Burnett equations did result in unstable equations (Bobylev's instability). Moreover, the Chapman-Enskog scaling is not appropriate for Knudsen layers, and thus the Burnett-type equations do not properly describe these. For a more detailed discussion, and for references on efforts to stabilize the Burnett equations, see [1].

Grad's moment method [2] leads to stable equations at higher orders and can describe Knudsen layers, but due to the hyperbolic character of the equations, shock structure calculations show spurious sub-shocks.

The biggest obstacle of the aforementioned approaches is the lack of suitable boundary conditions for the various sets of equations, which therefore cannot be applied to boundary value problems.

The regularized 13 moment (R13) equations are a macroscopic set of transport equations of super-Burnett order (i.e., of third order in the Knudsen number Kn) in the Chapman-Enskog sense that combine the benefits of Grad-type moment equations and Burnett-type equations, while omitting their problems.

Space does not permit to go into a deeper discussion of the equations, but we state their main features [3, 1].

The R13 equations

(a) are derived in a rational manner from the Boltzmann equation,
(b) are of third order in the Knudsen number,
(c) are linearly stable for initial and boundary value problems,
(d) contain the classical Burnett and super-Burnett equations asymptotically,
(e) predict phase speeds and damping of ultrasound waves in excellent agreement to experiments,
(f) give smooth shock structures for all Mach numbers, with good agreement to DSMC simulations for Ma<3,
(g) exhibit Knudsen boundary layers in good agreement to DSMC.
(h) are now furnished with a complete theory of boundary conditions [4], and
(j) obey an H-theorem for the linear case, including the boundary conditions [5].

With this, the R13 equations are the only extended hydrodynamic model at Burnett or super-Burnett order that is accessible to analytical and numerical solutions of boundary value problems. So far, the solutions show excellent agreement with DSMC simulations for Knudsen numbers below unity [4, 6]. In particular, the R13 equations exhibit typical rarefaction phenomena like temperature and velocity slip at boundaries, heat flux not driven by temperature gradient, anisotropic normal stresses, and Knudsen layers. Further simulations and comparison to solutions of the Boltzmann equation are in preparation.

In the present paper, we shall consider a reduced form of the R13 equations that does not include Knudsen layers. It will be seen that terms of super-Burnett order give rise to (a) second order jump conditions with numerical factors close to those found in the literature [7, 8, 9], and (b) to the characteristic dip in the temperature profile that is observed in force driven Poiseuille flow [10, 11, 12, 13].

CP1084, *Rarefied Gas Dynamics: 26th International Symposium,* edited by T. Abe

R13 EQUATIONS FOR CHANNELS

We consider plane channel flow between two infinite, parallel, resting plates in distance L, at steady state. The flow is driven by the specific body force G_1 which acts in the direction of the walls. All quantities in the equations below are made dimensionless with equilibrium pressure $p_0 = \rho_0\theta_0$, equilibrium temperature (in energy units) θ_0, and channel distance L. This leads to the occurrence of the the Knudsen number $\mathrm{Kn} = \frac{\mu_0\sqrt{\theta_0}}{p_0 L}$ where μ_0 is the viscosity at equilibrium conditions. In equilibrium the dimensionless moments assume the values $\rho = p = \theta = 1$, $\mu = 1$, $v = \sigma_{ij} = q_i = \Delta = R_{ij} = m_{ijk} = 0$.

Under the prescribed geometry, the R13 equations for monatomic ideal gases reduce to the following set of balance equations for dimensionless mass density ρ, velocity v, temperature θ, stress components $\sigma_{12}, \sigma_{11}, \sigma_{22}$ and heat flux components q_1, q_2:

$$\begin{aligned}
\frac{\partial \sigma_{12}}{\partial y} &= \rho G_1 \, , \\
\frac{\partial (p+\sigma_{22})}{\partial y} &= 0 \, , \\
\frac{\partial q_2}{\partial y} &= -\sigma_{12}\frac{\partial v}{\partial y} \\
\frac{8}{5}\sigma_{12}\frac{\partial v}{\partial y} + \frac{\partial m_{112}}{\partial y} &= -\frac{1}{\mathrm{Kn}}\frac{p}{\mu}\sigma_{11} \\
\frac{2}{5}\frac{\partial q_1}{\partial y} + P_0\frac{\partial v}{\partial y} + \frac{\partial m_{122}}{\partial y} &= -\frac{1}{\mathrm{Kn}}\frac{p}{\mu}\sigma_{12} \\
-\frac{6}{5}\sigma_{12}\frac{\partial v}{\partial y} + \frac{\partial m_{222}}{\partial y} &= -\frac{1}{\mathrm{Kn}}\frac{p}{\mu}\sigma_{22} \\
\frac{7}{5}q_2\frac{\partial v}{\partial y} + \frac{7}{2}\sigma_{12}\frac{\partial \theta}{\partial y} + \frac{1}{2}\frac{\partial R_{12}}{\partial y} + m_{112}\frac{\partial v}{\partial y} + \theta\frac{\partial \sigma_{12}}{\partial y} - \frac{\sigma_{11}}{\rho}\frac{\partial \sigma_{12}}{\partial y} &= -\frac{2}{3}\frac{1}{\mathrm{Kn}}\frac{p}{\mu}q_1 \\
\frac{5}{2}P_0\frac{\partial \theta}{\partial y} + \theta\frac{\partial \sigma_{22}}{\partial y} + \sigma_{22}\frac{\partial \theta}{\partial y} + \frac{2}{5}q_1\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial R_{22}}{\partial y} + \frac{1}{6}\frac{\partial \Delta}{\partial y} + m_{122}\frac{\partial v}{\partial y} - \frac{\sigma_{12}}{\rho}\frac{\partial \sigma_{12}}{\partial y} &= -\frac{2}{3}\frac{1}{\mathrm{Kn}}\frac{p}{\mu}q_2
\end{aligned}$$

The above equations contain the additional quantities Δ, R_{ij}, m_{ijk}, that vanish in Grad's classical 13 moment equations [2], while in the R13 theory they obey the constitutive laws [1]

$$\begin{aligned}
\Delta &= -\frac{2}{\rho}\left(\sigma_{12}^2+\sigma_{11}^2+\sigma_{22}^2+\sigma_{11}\sigma_{22}\right) - 12\mathrm{Kn}\frac{\mu}{p}\left[\theta\frac{\partial q_2}{\partial y} + \frac{7}{2}q_2\frac{\partial \theta}{\partial y} - \frac{q_2}{\rho}\frac{\partial p}{\partial y} + \theta\sigma_{12}\frac{\partial v}{\partial y}\right] \\
R_{12} &= -\frac{4}{7}\frac{1}{\rho}\sigma_{12}\left(\sigma_{11}+\sigma_{22}\right) - \frac{12}{5}\mathrm{Kn}\frac{\mu}{p}\left[\theta\frac{\partial q_1}{\partial y} + 2q_1\frac{\partial \theta}{\partial y} - \frac{q_1}{\rho}\frac{\partial p}{\partial y} + \frac{5}{7}\theta\left(\sigma_{11}+\sigma_{22}\right)\frac{\partial v}{\partial y}\right] \\
R_{22} &= -\frac{4}{21}\frac{1}{\rho}\left(\sigma_{12}^2+\sigma_{22}^2-2\sigma_{11}^2-2\sigma_{11}\sigma_{22}\right) - \frac{16}{5}\mathrm{Kn}\frac{\mu}{p}\left[\theta\frac{\partial q_2}{\partial y} + q_2\frac{\partial \theta}{\partial y} - \theta q_2\frac{\partial \ln\rho}{\partial y} + \frac{5}{14}\theta\sigma_{12}\frac{\partial v}{\partial y}\right] \\
m_{112} &= -\frac{2}{3}\mathrm{Kn}\frac{\mu}{p}\left[\theta\left(\frac{\partial \sigma_{11}}{\partial y} - \frac{2}{5}\frac{\partial \sigma_{22}}{\partial y}\right) + \left(\sigma_{11}-\frac{2}{5}\sigma_{22}\right)\frac{\partial \theta}{\partial y} - \left(\sigma_{11}-\frac{2}{5}\sigma_{22}\right)\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{16}{25}\frac{\partial v}{\partial y}q_1\right] \\
m_{122} &= -\frac{16}{15}\mathrm{Kn}\frac{\mu}{p}\left[\theta\frac{\partial \sigma_{12}}{\partial y} + \sigma_{12}\frac{\partial \theta}{\partial y} - \frac{\sigma_{12}}{\rho}\frac{\partial p}{\partial y} + \frac{2}{5}\frac{\partial v}{\partial y}q_2\right] \\
m_{222} &= -\frac{6}{5}\mathrm{Kn}\frac{\mu}{p}\left[\theta\frac{\partial \sigma_{22}}{\partial y} + \sigma_{22}\frac{\partial \theta}{\partial y} - \frac{\sigma_{22}}{\rho}\frac{\partial p}{\partial y} - \frac{4}{15}\frac{\partial v}{\partial y}q_1\right]
\end{aligned}$$

By Chapman-Enskog expansion it can be shown that Grad's 13 moment equations lead to the Burnett equations, while the R13 equations lead to the super-Burnett equations [1]. Therefore, the above equations for Δ, R_{ij}, m_{ijk} describe super-Burnett effects.

A full set of boundary conditions for the R13 equations, derived from the boundary conditions for the Boltzmann equation, is given in [4]. In the present context, where we will ignore Knudsen layers, we shall need only jump and slip boundary conditions, that will be discussed further below.

CHAPMAN-ENSKOG SCALING

We shall now reduce the above equations based on the Chapman-Enskog order of magnitude of the moments, in the same manner as presented in the last RGD meeting for Grad's moment equations [14]. We emphasize that this procedure will eliminate all Knudsen layer contributions from the equations. Detailed studies of the full R13 equations including the Knudsen layers are available in [1, 3, 4, 6].

The scales for the non-equilibrium quantities $\phi = \{\sigma_{ij}, \Delta, R_{ij}, m_{ijk}\}$ are obtained from a Chapman-Enskog expansion of the above equations. To this end we set $\phi = \phi_0 + \mathrm{Kn}\phi_1 + \mathrm{Kn}^2\phi_2 + \mathrm{Kn}^3\phi_3 + \cdots$, where ϕ_α are the expansion coefficients. We are not interested in the details of the expansion coefficients, but only in the leading order of the moments. A quantity ϕ is said to be of n-th order in Kn if the expansion coefficients at all orders below n vanish, $\phi_\alpha = 0$ for $\alpha < n$. A n-th order quantity is then rescaled as $\phi = \mathrm{Kn}^n\tilde{\phi}$ where the rescaled value $\tilde{\phi}$ is of order unity. This will make all scales in the equations explicit.

To simplify the procedure, we assume that G_1 scales like the Knudsen number,

$$G_1 = \mathrm{Kn}\tilde{G}_1 \ .$$

The equilibrium quantities temperature, density, and pressure are not expanded. Only shear stress and heat flux normal to the wall turn out to be first order quantities,

$$\sigma_{12} = \mathrm{Kn}\tilde{\sigma}_{12} \quad , \quad q_2 = \mathrm{Kn}\tilde{q}_2 \ . \tag{1}$$

Normal stresses, normal heat flux, and some of the higher moments are of second order,

$$\sigma_{11} = \mathrm{Kn}^2\tilde{\sigma}_{11} \ , \ \sigma_{22} = \mathrm{Kn}^2\tilde{\sigma}_{22} \ , \ q_1 = \mathrm{Kn}^2\tilde{q}_1 \ , \ \Delta = \mathrm{Kn}^2\tilde{\Delta} \ , \ R_{22} = \mathrm{Kn}^2\tilde{R}_{22} \ , \ m_{122} = \mathrm{Kn}^2\tilde{m}_{122} \ , \tag{2}$$

while the remaining non-equilibrium quantities are of third order,

$$R_{12} = \mathrm{Kn}^3\tilde{R}_{12} \ , \ m_{222} = \mathrm{Kn}^3\tilde{m}_{222} \ , \ m_{112} = \mathrm{Kn}^3\tilde{m}_{112} \ . \tag{3}$$

When the rescaled variables are introduced into the R13 equations, it is straightforward to reduce these so that only terms up to second order are retained. As an example we consider the equation for heat flux q_2 which in the rescaled variables assumes the form

$$\frac{5}{2}p\frac{\partial\theta}{\partial y} + \mathrm{Kn}^2\left[\frac{7}{2}\tilde{\sigma}_{22}\frac{\partial\theta}{\partial y} + \theta\frac{\partial\tilde{\sigma}_{22}}{\partial y} + \frac{2}{5}\tilde{q}_1\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial\tilde{R}_{22}}{\partial y} + \frac{1}{6}\frac{\partial\tilde{\Delta}}{\partial y} + \tilde{m}_{122}\frac{\partial v}{\partial y} - \frac{\tilde{\sigma}_{12}}{\rho}\frac{\partial\tilde{\sigma}_{12}}{\partial y}\right] = -\frac{2}{3}\frac{p}{\mu}\tilde{q}_2 \ . \tag{4}$$

Since q_2 is of first order in Kn, the expression $\mathrm{Kn}^2[\cdots]$ in the above equation for $\tilde{q}_2$ describes a third order correction to q_2 and can be ignored, so that the equation reduces to Fourier's law, $\tilde{q}_2 = -\frac{15}{4}\mu\frac{\partial\theta}{\partial y}$ to second order accuracy. We shall later also consider the third order terms in (4) which are responsible for the dip in the temperature profile.

To second order, our equations reduce to the conservation laws (P_0 is a constant of integration)

$$\frac{\partial\tilde{\sigma}_{12}}{\partial y} = \rho\tilde{G}_1 \ , \ p + \mathrm{Kn}^2\tilde{\sigma}_{22} = P_0 \ , \ \frac{\partial\tilde{q}_2}{\partial y} = -\tilde{\sigma}_{12}\frac{\partial v}{\partial y} \ , \tag{5}$$

the laws of Navier-Stokes and Fourier for shear stress and normal heat flux,

$$\tilde{\sigma}_{12} = -\mu\frac{\partial v}{\partial y} \ , \ \tilde{q}_2 = -\frac{15}{4}\mu\frac{\partial\theta}{\partial y} \ , \tag{6}$$

and the equations for second order quantities (2), which we can rewrite, using (6) as

$$\begin{aligned}
\tilde{\sigma}_{11} &= \frac{8}{5}\frac{\tilde{\sigma}_{12}\tilde{\sigma}_{12}}{p} \quad , \quad \tilde{\sigma}_{22} = -\frac{6}{5}\frac{\tilde{\sigma}_{12}\tilde{\sigma}_{12}}{p} \quad , \quad \tilde{q}_1 = -\frac{3}{2}\frac{\mu\theta}{p}\frac{\partial\tilde{\sigma}_{12}}{\partial y} + \frac{7}{2}\frac{\tilde{\sigma}_{12}\tilde{q}_2}{p} \ , \\
\tilde{\Delta} &= -12\frac{\mu\theta}{p}\frac{\partial\tilde{q}_2}{\partial y} + \frac{56}{5}\frac{\tilde{q}_2\tilde{q}_2}{p} + 10\theta\frac{\tilde{\sigma}_{12}\tilde{\sigma}_{12}}{p} \quad , \quad \tilde{R}_{22} = -\frac{16}{5}\frac{\mu\theta}{p}\frac{\partial\tilde{q}_2}{\partial y} + \frac{128}{75}\frac{\tilde{q}_2\tilde{q}_2}{p} + \frac{20}{21}\frac{\theta}{p}\tilde{\sigma}_{12}\tilde{\sigma}_{12} \quad , \\
\tilde{m}_{122} &= -\frac{16}{15}\frac{\mu\theta}{p}\frac{\partial\tilde{\sigma}_{12}}{\partial y} + \frac{32}{45}\frac{\tilde{\sigma}_{12}\tilde{q}_2}{p} \ .
\end{aligned} \tag{7}$$

Equations for the third order quantities (3) will not be required further, and are not shown.

We repeat that the R13 equations describe Knudsen layers [4, 5, 6], which do not obey Chapman-Enskog scaling. The above reduction removes the Knudsen layer solutions. Thus, the above equations are valid in the bulk, or when Knudsen layers can be ignored, in particular in strongly non-linear flow regimes.

2ND ORDER JUMP AND SLIP

In the previous section it was shown that even to second order the R13 equations reduce to the conservation laws for momentum and energy with the Navier-Stokes and Fourier laws. As we shall see now, super-Burnett effects come into play through the boundary conditions. For the reduced equations (5, 6) we require boundary conditions for velocity slip and temperature jump, which were derived in [4] from Maxwell's boundary conditions for the Boltzmann equation in terms of the moments as

$$\begin{aligned}\sigma_{12} &= -\frac{\chi_1}{2-\chi_1}\sqrt{\frac{2}{\pi\theta}}\left[\mathscr{P}(v-v_W)+\frac{1}{5}q_1+\frac{1}{2}m_{122}\right]n_2\,, \qquad (8)\\ q_2 &= -\frac{\chi_2}{2-\chi_2}\sqrt{\frac{2}{\pi\theta}}\left[2\mathscr{P}(\theta-\theta_W)-\frac{1}{2}\mathscr{P}V^2+\frac{1}{2}\theta\sigma_{22}+\frac{1}{15}\Delta+\frac{5}{28}R_{22}\right]n_2\,.\end{aligned}$$

Here, v_W and θ_W are the velocity and temperature of the wall, χ_1 and χ_2 are accommodation coefficients for shear stress and heat flux, and $\mathscr{P}=\rho\theta+\frac{1}{2}\sigma_{22}-\frac{1}{120}\frac{\Delta}{\theta}-\frac{1}{28}\frac{R_{22}}{\theta}$; n_2 is the wall normal, pointing into the gas.

To approximate the boundary conditions up to second order in Kn, we introduce the rescaled quantities $\tilde{\phi}$ into the boundary conditions. For consistent scaling, velocity slip and temperature jump must be rescaled as

$$v-v_W=\mathrm{Kn}\mathscr{V}\ ,\ \ \theta-\theta_W=\mathrm{Kn}\mathscr{T}\,.$$

Then, it is straightforward to remove higher order terms and obtain the second order jump and slip conditions as

$$\begin{aligned}\mathscr{V} &= -\frac{2-\chi_1}{\chi_1}\sqrt{\frac{\pi\theta}{2}}\frac{\tilde{\sigma}_{12}}{p}n_2-\frac{1}{5}\mathrm{Kn}\frac{\tilde{q}_1}{p}-\frac{1}{2}\mathrm{Kn}\frac{\tilde{m}_{122}}{p}\,,\\ \mathscr{T} &= -\frac{2-\chi_2}{\chi_2}\sqrt{\frac{\pi\theta}{2}}\frac{\tilde{q}_2}{2p}n_2+\frac{1}{4}\mathrm{Kn}\mathscr{V}^2-\frac{1}{4}\theta\mathrm{Kn}\frac{\tilde{\sigma}_{22}}{p}-\frac{1}{60}\mathrm{Kn}\frac{\tilde{\Delta}}{p}-\frac{5}{56}\mathrm{Kn}\frac{\tilde{R}_{22}}{p}\,.\end{aligned}$$

The second order boundary conditions require contributions from the higher moments $\tilde{m}_{122}$, $\tilde{\Delta}$, $\tilde{R}_{22}$ which contribute terms of super-Burnett order to the transport equations, but terms of Burnett order to the boundary conditions.

Now we can insert our constitutive equations for higher moments (7), and also use the energy balance $(5)_3$ $\frac{\partial\tilde{q}_2}{\partial y}=\tilde{\sigma}_{12}^2/\mu$ to replace $\tilde{\sigma}_{12}^2/\mu$ in the jump equation to obtain

$$\mathscr{V} = -\frac{2-\chi_1}{\chi_1}\sqrt{\frac{\pi\theta}{2}}\frac{\tilde{\sigma}_{12}}{p}n_2+\frac{5}{6}\mathrm{Kn}\frac{\mu\theta}{p^2}\frac{\partial\tilde{\sigma}_{12}}{\partial y}-\frac{19}{18}\mathrm{Kn}\frac{\tilde{\sigma}_{12}\tilde{q}_2}{p^2}\,, \qquad (9)$$

$$\mathscr{T} = -\frac{2-\chi_2}{\chi_2}\sqrt{\frac{\pi\theta}{2}}\frac{\tilde{q}_2}{2p}n_2+\mathrm{Kn}\left[\frac{\pi}{8}\frac{1}{\beta_1^2}+\frac{157}{294}\right]\frac{\mu\theta}{p^2}\frac{\partial\tilde{q}_2}{\partial y}-\mathrm{Kn}\frac{178}{525}\frac{\tilde{q}_2\tilde{q}_2}{p^2}\,. \qquad (10)$$

Second order slip and jump conditions are widely available in the literature. For comparison we focus on the slip condition (9), in which we introduce the Navier-Stokes and Fourier laws, so that

$$v-v_W=\frac{2-\chi_1}{\chi_1}\mathrm{Kn}\sqrt{\frac{\pi\theta}{2}}\frac{\mu}{p}\frac{\partial v}{\partial y}n_2-\frac{5}{6}\mathrm{Kn}^2\frac{\mu^2\theta}{p^2}\frac{\partial^2 v}{\partial y^2}-\mathrm{Kn}^2\frac{\mu^2}{p^2}\left[\frac{5}{6}\frac{d\ln\mu}{d\ln\theta}+\frac{95}{24}\mathrm{Kn}^2\frac{\mu^2}{p^2}\right]\frac{\partial\theta}{\partial y}\frac{\partial v}{\partial y}\,.$$

For slow flows, one might ignore the non-linear term, so that

$$v-v_W=\alpha\mathrm{Kn}\sqrt{\frac{\pi\theta}{2}}\frac{\mu}{p}\frac{\partial v}{\partial y}n_2-\beta\mathrm{Kn}^2\frac{\mu^2\theta}{p^2}\frac{\partial^2 v}{\partial y^2}\,. \qquad (11)$$

with $\alpha=\frac{2-\chi_1}{\chi_1}$ and $\beta=\frac{5}{6}$. The following table shows values for the second order slip coefficient β taken from different authors for the case of full accommodation ($\chi_1=1$),

R13 [5]	Deissler [7]	Hadjiconstantinou [8]	Lockerby et al. [9]
$\frac{5}{6}=0.833$	$\frac{9\pi}{16}=1.767$	$0.606\frac{\pi}{2}=0.952$	$\frac{3}{10}=0.3$

We see that the value for β from the R13 equations is quite close to the value by Hadjiconstantinou. Deisslers value seems a bit high. Lockerby et al. only used the Burnett equation for q_1 in (8), and thus missed the super-Burnett contribution of m_{122}. For the case of full accommodation ($\chi_1=1$), all cited authors report a value $\alpha=1$, only Hadjiconstantinou reports a corrected slip coefficient $\alpha_H=1.1466$.

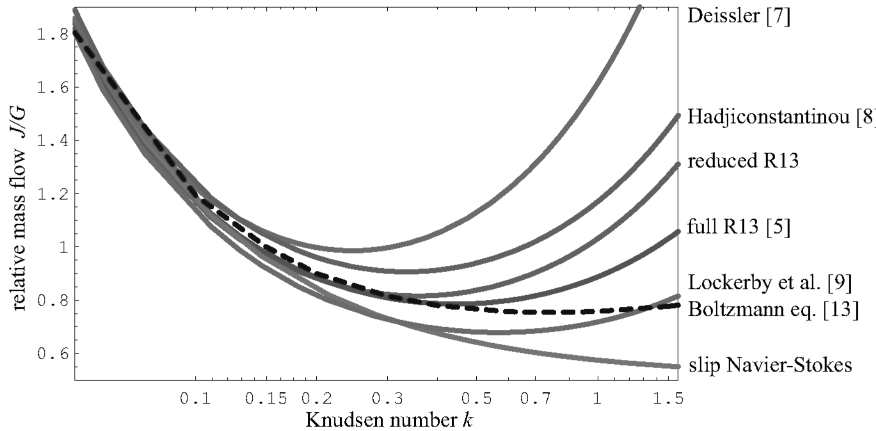

FIGURE 1. Relative mass flow rate J/G_1 over Ohwada's Knudsen number $k = \frac{4}{5}\sqrt{2}\mathrm{Kn}$, for Boltzmann equation (dashed) [15], full R13 equations [5], the reduced R13 equations of the present paper, and Navier-Stokes with second order slip conditions from Deissler [7], Hadjiconstantinou [8], Lockerby et al. [9]

APPROXIMATE SOLUTION

In order to study the 2nd order slip effects in a simple manner, we ignore variations of density and viscosity which are assumed to have their equilibrium values (i.e., unity), so that the stress and velocity follow from

$$\frac{\partial \tilde{\sigma}_{12}}{\partial y} = \tilde{G}_1 \ , \quad \tilde{\sigma}_{12} = -\frac{\partial v}{\partial y}$$

with the boundary condition (11) ($v_W = 0$, resting walls at $y = \pm\frac{1}{2}$), as, with $\tilde{G}_1 = G_1/\mathrm{Kn}$,

$$v = \frac{G_1}{2\mathrm{Kn}} \left[\left(\frac{1}{4} - y^2 \right) + \alpha \mathrm{Kn} \sqrt{\frac{\pi}{2}} + 2\beta \, \mathrm{Kn}^2 \right] \ , \ \sigma_{12} = -G_1 y \, .$$

The average mass flow rate is

$$J = \int_{-1/2}^{1/2} v dy = \frac{G_1}{12\mathrm{Kn}} \left[1 + 6\alpha \mathrm{Kn} \sqrt{\frac{\pi}{2}} + 12\beta \, \mathrm{Kn}^2 \right] \, .$$

Figure 1 compares the reduced mass flow J/G_1 computed from higher order theories to the numerical solution of the Boltzmann equation by Ohwada et al. [15]. Clearly, the solution of the full R13 equations—which includes Knudsen layers—in [5] gives the best match, while the reduced R13 equations give numerical values for the second order slip coefficient that lead to the best match with the full Boltzmann solution.

Linearized in θ, p, and μ the first law reduces to

$$\frac{\partial \tilde{q}_2}{\partial y} = \tilde{\sigma}_{12}^2 = \tilde{G}_1^2 y^2 \quad \text{so that} \quad \tilde{q}_2 = \frac{1}{3} \tilde{G}_1^2 y^3$$

where the integration constant is zero due to symmetry. Within the same approximation, and considering only terms up to order G^3 in the driving force, we find for the higher moments (7)

$$\tilde{\sigma}_{22} = -\frac{6}{5}\tilde{G}_1^2 y^2 \ , \ \tilde{q}_1 \simeq \frac{3}{2}\tilde{G}_1 - \frac{7}{6}\tilde{G}_1^3 y^4 \ , \ \tilde{R}_{22} \simeq -\frac{236}{105}\tilde{G}_1^2 y^2 \ , \ \tilde{\Delta} = -2\tilde{G}_1^2 y^2 \ , \ \tilde{m}_{122} = -\frac{16}{15}\tilde{G}_1 - \frac{32}{135}\tilde{G}_1^3 y^4$$

Now, we consider the equation for heat flux (4) where we include the *third* order contributions (the factor on Kn^2), to find, again, up to terms in G_1^3,

$$\frac{\partial \theta}{\partial y} = -\frac{4}{45}\tilde{G}_1^2 y^3 + \frac{976}{525}\mathrm{Kn}^2 \tilde{G}_1^2 y$$

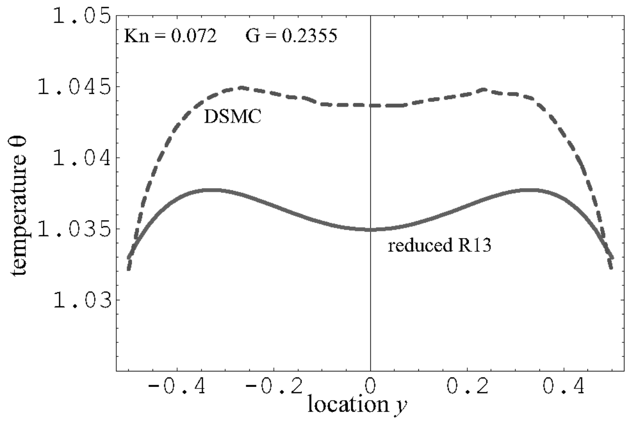

FIGURE 2. Temperature profile for force driven channel flow, DSMC result [13] and reduced R13 result, Eq. (12).

Integration, and use of the jump boundary conditions (10) gives, with $\tilde{G}_1 = \frac{G_1}{\mathrm{Kn}}$,

$$\theta - \theta_W = \frac{G_1^2}{8\mathrm{Kn}}\left[\sqrt{\frac{\pi}{2}}\frac{1}{6} + \mathrm{Kn}\left(\frac{\pi}{4} + \frac{157}{147}\right)\right] + \frac{G_1^2}{\mathrm{Kn}^2}\left[\frac{1}{45}\left(\frac{1}{16} - y^4\right) - \frac{488}{525}\mathrm{Kn}^2\left(\frac{1}{4} - y^2\right)\right]. \quad (12)$$

Here, the first term describes the temperature jump at the wall, while the second contribution describes the temperature profile. The competition between the positive hydrodynamic term, $\frac{1}{45}\left(\frac{1}{16} - y^4\right)$, and the negative R13 correction, $-\frac{488}{525}\mathrm{Kn}^2\left(\frac{1}{4} - y^2\right)$, leads to the characteristic dip of the temperature profile that was reported in [12]. Figure 2 compares the temperature curve of Eq. (12) to DSMC data from Ref. [13] for $\mathrm{Kn} = 0.072$, $G_1 = 0.2355$. The main difference between the two curves are the more pronounced shoulders at the walls in the DSMC simulation, which are due to Knudsen layers that were ignored in the present simplified analysis. Our analytic computation of the temperature profile stands in agreement with the results of Ref. [10, 11, 13], who showed that the characteristic dip is a super-Burnett effect. An analytical solution that includes the Knudsen layers is presented in [6].

ACKNOWLEDGEMENT

This research was supported by the Natural Sciences and Engineering Research Council (NSERC). The author likes to thank Dr. Kun Xu (Hong Kong) for providing the DSMC data for Poiseuille flow.

REFERENCES

1. H. Struchtrup, *Macroscopic Transport Equations for Rarefied Gas Flows—Approximation Methods in Kinetic Theory*. Interaction of Mechanics and Mathematics Series, Springer, Heidelberg 2005
2. H. Grad, Comm. Pure Appl. Math. **2**, 325 (1949)
3. Struchtrup, H. and Torrilhon, M., Phys. Fluids **15**, 2668-2680 (2003)
4. M. Torrilhon, H. Struchtrup, J. Comp. Phys. **227** 1982-2011(2008)
5. H. Struchtrup, M. Torrilhon, Phys. Rev. Lett. **99**, 014502 (2007)
6. P. Taheri, M. Torrilhon, H. Struchtrup, Couette and Poiseuille Flows in Microchannels, submitted to Phys. Fluids (2008)
7. R.G. Deissler, Int. J. Heat Mass Transfer **7**, 681-694 (1964)
8. N.G. Hadjiconstantinou, Phys. Fluids **15**, 2352-2354 (2003)
9. D. A. Lockerby, J. M. Reese, D. R. Emerson, R. W. Barber, Phys. Rev. E **70** 017303 (2004)
10. M. Tij, A. Santos, J. Stat. Phys. **76**, 1399–1414 (1994)
11. M. Tij, M. Sabbane, A. Santos, Phys. Fluids 10, 1021–1927 (1998)
12. Y. Zheng, A.L. Garcia, B.J. Alder, J. Stat. Phys. **109**, 495-505 (2002)
13. Kun Xu, Phys. Fluids **15**, 2077-2080 (2003)
14. H. Struchtrup, T. Thatcher, M. Torrilhon: in 25th International Symposium on Rarefied Gas Dynamics, M.S. Ivanov, A.K. Rebrov (Eds.), Publishing House of the Siberian Branch of the Russian Academy of Sciences, 91-96 (2007)
15. T. Ohwada, Y. Sone, and K. Aoki, Phys. Fluids A **1**, 2042-2049 (1989)