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ORIGINAL ARTICLE

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Bulk equations and Knudsen layers for the regularized 13 moment equations

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Abstract The order of magnitude method offers an alternative to the Chapman-Enskog and Grad methods to derive macroscopic transport equations for rarefied gas flows. This method yields the regularized 13 moment equations (R13) and a generalization of Grad's 13 moment equations for non-Maxwellian molecules. Both sets of equations are presented and discussed. Solutions of these systems of equations are considered for steady state Couette flow. The order of magnitude method is used to further reduce the generalized Grad equations to the non-linear bulk equations, which are of second order in the Knudsen number. Knudsen layers result from the linearized R13 equations, which are of the third order. Superpositions of bulk solutions and Knudsen layers show good agreement with DSMC calculations for Knudsen numbers up to 0.5.

Keywords Rarefied gas lows · Knudsen number expansion · Couette flow

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1 Introduction

Processes in rarefied gases are well described by the Boltzmann equation [3,4,8,18], a non-linear integro-differential equation that describes the evolution of the particle distribution function f in phase space, i.e., on the microscopic level.

The most important scaling parameter to characterize processes in rarefied gases is the Knudsen number Kn, defined as the ratio between the mean free path of a particle and a relevant reference length scale (e.g., channel width, wavelength). If the Knudsen number is small, the Boltzmann equation can be reduced to simpler models, which allow faster solutions. Indeed, if $\mathrm{Kn} < 0.01$ (say), the equations of ordinary hydrodynamics—the laws of Navier–Stokes and Fourier (NSF)—can be derived from the Boltzmann equation, e.g., by the Chapman–Enskog method [3,4,8,18]. The NSF equations are macroscopic equations for mass density ρ, velocity v_i and temperature T, and thus pose a mathematically less complex problem than the Boltzmann equation.

Macroscopic equations for rarefied gas flows at Knudsen numbers above 0.01 are highly desirable, since they promise to replace the Boltzmann equation with simpler equations that still capture the relevant physics. The Chapman–Enskog expansion is the classical method to achieve this goal, but the resulting Burnett and super-Burnett equations are unstable [1,13]. Thus, one either has to find ways to stabilize the Burnett and super-Burnett equations [2,5,23], or to find alternatives to the Chapman–Enskog expansion.

A classical alternative is Grad's moment method [6], which extends the set of variables beyond the hydrodynamic ones (ρ, v_i, T) by adding deviatoric stress tensor σ_{ij}, heat flux q_i, and possibly higher moments of

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the phase density. The resulting equations are stable, but lead to spurious discontinuities in shocks, and for given value of the Knudsen number it is not clear what set of moments one would have to consider [18].

Combinations of the Grad and Chapman–Enskog methods were first presented by Reinecke and Kremer who re-derived the Burnett equations [12]. Struchtrup and Torrilhon performed a Chapman–Enskog expansion around a non-equilibrium phase density of Grad type [15,18,22], see also [6,7] for earlier attempts. This approach led to the "regularized 13 moment equations" (R13 equations) which form a stable set of equations for the 13 variables ρ, v_i, T, σ_{ij}, q_i at super-Burnett order.

A weak point in the Grad method is that no statement is made to connect the Knudsen number and relevant moments. As a consequence the original derivation of the R13 equations in [15,22] required the assumption of different time scales for the basic 13 moments, and higher moments. While this assumption leads to a set of equations with desired behavior, it is difficult to justify since the characteristic times of all moments are, in fact, of the same order.

An alternative approach to the problem was presented by Struchtrup in Refs. [16–18], partly based on earlier work by Müller et al. [11]. The *Order of Magnitude Method* derives approximations to the Boltzmann equation from its infinite set of corresponding moment equations [16–18]. This method first determines the order of magnitude of all moments by means of a Chapman–Enskog expansion, forms linear combinations of moments in order to have the minimum number of moments at a given order, and then uses the information on the order of the moments to properly rescale the moment equations. Finally, the rescaled moment equations are systematically reduced by cancelling terms of higher order.

The order of magnitude method gives the Euler and NSF equations at zeroth and first order, and thus agrees with the Chapman–Enskog method in the lower orders [16]. The second-order equations turn out to be Grad's 13 moment equations [6] for Maxwell molecules, and a generalization of these for molecules that interact with power potentials [18]. At the third order, the method was only performed for Maxwell molecules, where it yields the R13 equations [16].

In the remainder of this paper an approximate solution of the R13 equations for plane Couette flow is constructed. The geometry of the problem allows one to further reduce the transport equations. The order of magnitude method is used to derive a non-linear system of equations of second order in Knudsen number from the generalized Grad equations. These *bulk equations* describe the flow outside the Knudsen layer. Knudsen layer contributions are computed from the linearized R13 equations. The presented solutions are superpositions of the numerical solution of the bulk equations and the Knudsen layers. This approach is similar to that presented in [9]. For moderate Knudsen numbers, the results stand in excellent agreement to DSMC simulations.

2 Order of magnitude method

The *order of magnitude method* considers not the Boltzmann equation itself, but its infinite system of moment equations. The method of finding the proper equations with *order of accuracy* λ_0 in the Knudsen number consists of the following three steps:

1. Determination of the *order of magnitude* λ of the moments.
2. Construction of a moment set with minimum number of moments at any order λ.
3. Deletion of all terms in all equations that would lead only to contributions of orders $\lambda > \lambda_0$ in the conservation laws for energy and momentum.

Step 1 is based on a Chapman–Enskog expansion where a moment ϕ is expanded according to

$$\phi = \phi_0 + \mathrm{Kn}\phi_1 + \mathrm{Kn}^2\phi_2 + \mathrm{Kn}^3\phi_3 + \cdots , \tag{1}$$

and the leading order of ϕ is determined by inserting this ansatz into the complete set of moment equations. A moment is said to be of leading order λ if $\phi_\beta = 0$ for all $\beta < \lambda$. This first step agrees with the ideas of Ref. [11]. Alternatively, the order of magnitude of the moments can be found from the principle that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms* [20].

In Step 2 new variables are introduced by linear combination of the moments originally chosen. The new variables are constructed such that the number of moments at a given order λ is minimal. This step does not only simplify the later discussion, but gives an unambiguous set of moments at order λ. This ensures that the final result will be independent of the initial choice of moments.

Step 3 follows from the definition of the order of accuracy λ_0: A set of equations is said to be accurate of order λ_0, when stress σ_{ij} and heat flux q_i are known within the order $\mathcal{O}\left(\mathrm{Kn}^{\lambda_0}\right)$. The evaluation of this condition is based on the fact that all moment equations are coupled. This implies that each term in any of the moment equations has some influence on all other equations, in particular on the conservation laws. A theory of order λ_0 will consider only those terms in all equations whose leading order of *influence* in the conservation laws is $\lambda \leq \lambda_0$. Luckily, in order to evaluate this condition, it suffices to start with the conservation laws, and step by step, order by order, add the relevant terms that are required

The order of magnitude method was applied to the special cases of Maxwell molecules and the BGK model in Refs. [16,18], and it was shown that it yields the Euler equations at zeroth order, the NSF equations at first order, and Grad's 13 moment equations (with omission of the non-linear term $\frac{\sigma_{ij}}{\rho}\frac{\partial\sigma_{jk}}{\partial x_k}$) at the second order. The regularized 13 moment equations (R13) are obtained as the third-order approximation. They consist of the conservation laws for mass, momentum and energy (with $\theta = RT$, and the ideal gas law $p = \rho\theta$; R is the gas constant),

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} &= 0\ , \\ \rho\frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{\partial x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ik}}{\partial x_k} &= 0\ , \\ \frac{3}{2}\rho\frac{\partial \theta}{\partial t} + \frac{3}{2}\rho v_k \frac{\partial \theta}{\partial x_k} + \frac{\partial q_k}{\partial x_k} &= -\left(p\delta_{ij} + \sigma_{ij}\right)\frac{\partial v_i}{\partial x_j}\ , \end{aligned} \tag{2}$$

and the balance laws for deviatoric stress σ_{ij} (with $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{kk} = 0$) and heat flux q_k, which read

$$\frac{\partial \sigma_{ij}}{\partial t} + v_k\frac{\partial \sigma_{ij}}{\partial x_k} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + \frac{\partial m_{ijk}}{\partial x_k} = -\frac{p}{\mu}\sigma_{ij}\ , \tag{3}$$

$$\begin{aligned} \frac{\partial q_i}{\partial t} + v_k\frac{\partial q_i}{\partial x_k} + \frac{5}{2}p\frac{\partial \theta}{\partial x_i} + \frac{5}{2}\sigma_{ik}\frac{\partial \theta}{\partial x_k} + \theta\frac{\partial \sigma_{ik}}{\partial x_k} - \theta\sigma_{ik}\frac{\partial \ln\rho}{\partial x_k} + \frac{7}{5}q_k\frac{\partial v_i}{\partial x_k} + \frac{2}{5}q_k\frac{\partial v_k}{\partial x_i} \\ + \frac{7}{5}q_i\frac{\partial v_k}{\partial x_k} + \frac{1}{2}\frac{\partial R_{ik}}{\partial x_k} + \frac{1}{6}\frac{\partial \Delta}{\partial x_i} + m_{ijk}\frac{\partial v_j}{\partial x_k} - \frac{\sigma_{ij}}{\rho}\frac{\partial \sigma_{jk}}{\partial x_k} = -\frac{2}{3}\frac{p}{\mu}q_i\ . \end{aligned} \tag{4}$$

Here, the additional quantities m_{ijk}, R_{ij} and Δ are moments of higher order [15]. For second-order accuracy, these equations are closed by setting

$$\Delta = R_{ij} = m_{ijk} = 0\ . \tag{5}$$

The resulting equations are Grad's original 13 moment equations. However, from the order of magnitude argument the term $\left(\frac{\sigma_{ij}}{\rho}\frac{\partial\sigma_{jk}}{\partial x_k}\right)$ turns to be of the third order, this term can be neglected in a second-order theory.

For third-order accuracy, the equations are closed by the expressions [16]

$$\begin{aligned} \Delta &= -\frac{\sigma_{ij}\sigma_{ij}}{\rho} - 12\frac{\mu}{p}\left[\theta\frac{\partial q_k}{\partial x_k} + \theta\sigma_{kl}\frac{\partial v_k}{\partial x_l} + \frac{5}{2}q_k\frac{\partial \theta}{\partial x_k} - q_k\theta\frac{\partial \ln\rho}{\partial x_k}\right]\ , \\ R_{ij} &= -\frac{4}{7}\frac{1}{\rho}\sigma_{k\langle i}\sigma_{j\rangle k} - \frac{24}{5}\frac{\mu}{p}\left[\theta\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + q_{\langle i}\frac{\partial \theta}{\partial x_{j\rangle}} - \theta q_{\langle i}\frac{\partial \ln\rho}{\partial x_{j\rangle}} + \frac{10}{7}\theta\sigma_{k\langle i}\frac{\partial v_{\langle j\rangle}}{\partial x_{k\rangle}}\right]\ , \\ m_{ijk} &= -2\frac{\mu}{p}\left[\theta\frac{\partial \sigma_{\langle ij}}{\partial x_{k\rangle}} - \sigma_{\langle ij}\theta\frac{\partial \ln\rho}{\partial x_{k\rangle}} + \frac{4}{5}q_{\langle i}\frac{\partial v_j}{\partial x_{k\rangle}}\right]\ . \end{aligned} \tag{6}$$

The equations (2–4, 6) form the R13 equations.

In the above equations the indices between angular brackets refer to the symmetric and trace-free parts of tensors. μ denotes the viscosity with $\mu = \mu_0\left(\theta/\theta_0\right)^{\omega}$; for Maxwell molecules $\omega = 1$.

For general, i.e., non-Maxwellian, molecule types the order of magnitude method was performed to second order in Refs. [17,18]; the derivation of the third-order equations would be far more involved than for Maxwell molecules. Again the equations at zeroth and first order are the Euler and NSF equations with exact viscosity,

Table 1 Burnett coefficients for power potentials ($\gamma = 5$ for Maxwell molecules, $\gamma = \infty$ for hard sphere molecules) [12]

γ	ω	ϖ_1	ϖ_2	ϖ_3	ϖ_4	ϖ_5	ϖ_6	θ_1	θ_2	θ_3	θ_4	θ_5
5	1	10/3	2	3	0	3	8	75/8	45/8	−3	3	39/4
7	0.833	3.561	2.003	2.793	0.217	1.942	7.781	10.038	5.647	−3.010	2.793	9.113
7.66	0.8	3.600	2.004	2.761	0.254	1.784	7.748	10.160	5.656	−3.014	2.761	9.019
9	0.75	3.679	2.007	2.695	0.328	1.466	7.681	10.402	5.674	−3.023	2.695	8.829
17	0.625	3.863	2.016	2.553	0.500	0.814	7.543	10.995	5.736	−3.053	2.553	8.442
∞	0.5	4.056	2.028	2.418	0.681	0.219	7.424	11.644	5.822	−3.09	2.418	8.286

heat conductivity and Prandtl number. The second-order equations are a generalization of Grad's 13 moment equations,

$$\frac{\partial \sigma_{ij}}{\partial t} + v_k \frac{\partial \sigma_{ij}}{\partial x_k} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + \frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_3}{\varpi_2}\left(\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - \omega q_{\langle i}\frac{\partial \ln\theta}{\partial x_{j\rangle}}\right) + \frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_4}{\varpi_2} q_{\langle i}\frac{\partial \ln p}{\partial x_{j\rangle}}$$
$$+ \frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_5}{\varpi_2} q_{\langle i}\frac{\partial \ln\theta}{\partial x_{j\rangle}} + \left(\frac{\varpi_6}{\varpi_2} - 4\right)\sigma_{k\langle i}S_{j\rangle k} = -\frac{2}{\varpi_2}\frac{p}{\mu}\left[\sigma_{ij} + 2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\right], \tag{7}$$

$$\frac{\partial q_i}{\partial t} + v_k\frac{\partial q_i}{\partial x_k} + q_k\frac{\partial v_i}{\partial x_k} + \frac{5}{3}q_i\frac{\partial v_k}{\partial x_k} - \frac{5}{2}\frac{1}{\mathrm{Pr}}\sigma_{ik}\frac{\partial\theta}{\partial x_k} + \frac{5}{4}\frac{1}{\mathrm{Pr}}\frac{\theta_3}{\theta_2}\theta\sigma_{ik}\frac{\partial \ln p}{\partial x_k}$$
$$+ \frac{5}{4}\frac{1}{\mathrm{Pr}}\frac{\theta_4}{\theta_2}\theta\left(\frac{\partial \sigma_{ik}}{\partial x_k} - \omega\sigma_{ik}\frac{\partial\ln\theta}{\partial x_k}\right) + \frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{3}{2}\frac{\theta_5}{\theta_2}\sigma_{ik}\frac{\partial\theta}{\partial x_k} = -\frac{1}{\theta_2}\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{p}{\mu}\left[q_i + \frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{\partial\theta}{\partial x_i}\right]. \tag{8}$$

Here, ϖ_α, θ_α denote the Burnett coefficients as given in Table 1, $\mathrm{Pr} \simeq \frac{2}{3}$ is the Prandtl number, and ω is the viscosity exponent.

The order of magnitude method reproduces the established results of the Chapman–Enskog expansion at the zeroth (Euler) and first (NSF) order. Moreover it provides a new link between the Knudsen number and Grad's 13 moment equations which turn out to be of second order in the Knudsen number, together with a generalization of these for non-Maxwellian molecules. Finally, the method provides a rational derivation of the R13 equations.

The R13 equations contain the Burnett and super-Burnett equations when expanded in a series in the Knudsen number [15,22]. However, other than the Burnett and super-Burnett equations, the R13 equations are linearly stable for all wavelengths and frequencies [15]. Dispersion relation and damping for the R13 equations agree better with experimental data than those for the Navier–Stokes–Fourier equations, or the original 13 moments system [15]. They also allow the approximate description of Knudsen boundary layers [15,19]. Note that the accurate description of Knudsen layers requires the solution of the linearized Boltzmann equation [8].

3 Bulk equations for Couette geometry

The scaling processes of Chapman–Enskog expansion and of the order of magnitude method are applied to the complete set of equations without further consideration of geometry, boundary conditions, etc. Thus the basic scaling argument gives the maximum size (or minimum order of magnitude) of the moments involved. However, due to the particular geometry of a problem, the individual vector and tensor components of σ_{ij} or q_i can have different orders of magnitude. This allows one to further reduce the equations.

In this section the generalized 13 moment equations will be further reduced for steady-state Couette flow: Two infinite parallel plates at constant distance L move with the constant velocities v_W^0, v_W^L relative to each other in their respective planes, and are kept at constant temperatures θ_W^0, θ_W^L. The coordinate frame is chosen such that the planes move into the direction $x = x_1$, and $y = x_2$ is the direction perpendicular to the plates.

We are interested only in the second-order equations, and thus it is sufficient to base the argument on the generalized Grad equations (which are of the second order); a similar treatment of the R13 equations would lead to the same result (for Maxwell molecules).

3.1 Generalized 13 moment equations for Couette flow

Due to the symmetry of the problem, all variables will depend only on the coordinate y. Since the walls are impermeable, the velocity of the gas must point into the x-direction, that is

$$v_i = \{v(y), 0, 0\}_i \quad \text{and thus} \quad \frac{\partial v_k}{\partial x_k} = 0, \quad v_k \frac{\partial}{\partial x_k} = 0 \; . \tag{9}$$

Furthermore, since the setup is independent of the third space coordinate, $z = x_3$, neither stress nor heat flux should be associated with that direction, so that $\sigma_{13} = \sigma_{23} = q_3 = 0$, and

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11}(y) & \sigma_{12}(y) & 0 \\ \sigma_{12}(y) & \sigma_{22}(y) & 0 \\ 0 & 0 & -\sigma_{11}(y) - \sigma_{22}(y) \end{bmatrix}_{ij}, \quad q_i = \{q_1(y), q_2(y), 0\}_i \; . \tag{10}$$

The conservation laws (2) reduce to

$$0 = 0 \; , \; \frac{d\sigma_{12}}{dy} = 0, \quad \frac{d\rho\theta}{dy} + \frac{d\sigma_{22}}{dy} = 0, \quad \frac{d\sigma_{32}}{dy} = 0, \quad \frac{dq_2}{dy} = -\sigma_{12}\frac{dv}{dy}. \tag{11}$$

The first and fourth equation are trivial, and two of the remaining three can be integrated so that

$$\sigma_{12} = \sigma_{12}^0 = \text{const.}, \quad \rho\theta + \sigma_{22} = P_0 = \text{const.}, \quad \frac{dq_2}{dy} = -\sigma_{12}\frac{dv}{dy}. \tag{12}$$

The Eqs. (7, 8) for the relevant components of stress and heat flux can then be cast to read

$$\begin{aligned} &\left(\frac{\varpi_6}{16} + \frac{\varpi_2}{4} - 1\right)\sigma_{22}\frac{dv}{dy} - \frac{\mu}{p}\left(\frac{\varpi_6}{8} - \frac{\varpi_2}{2}\right)\sigma_{12}\frac{dv}{dy}\frac{dv}{dy} + \frac{1}{5}\Pr\left(\varpi_5 - \varpi_3\omega\right)q_1\frac{d\ln\theta}{dy} \\ &\quad - \frac{\varpi_4}{5}\Pr\frac{q_1}{P_0 - \sigma_{22}}\frac{d\sigma_{22}}{dy} + \frac{\varpi_3}{5}\Pr\frac{dq_1}{dy} + P_0\frac{\partial v}{\partial y} = -\frac{P_0 - \sigma_{22}}{\mu}\sigma_{12} \; , \end{aligned} \tag{13}$$

$$\begin{aligned} &\frac{4}{15}\Pr\left(\varpi_5 - \varpi_3\omega\right)q_2\frac{d\ln\theta}{dy} - \frac{4}{15}\Pr\varpi_4\frac{q_2}{P_0 - \sigma_{22}}\frac{d\sigma_{22}}{dy} \\ &\quad + \left(\frac{\varpi_6}{12} - \frac{2}{3}\varpi_2 - \frac{4}{15}\Pr\varpi_3\right)\sigma_{12}\frac{dv}{dy} = -\frac{P_0 - \sigma_{22}}{\mu}\sigma_{22} \; , \end{aligned} \tag{14}$$

$$-\frac{\theta_3}{2}\frac{\theta\sigma_{12}}{p}\frac{d\sigma_{22}}{dy} + \frac{2}{5}\Pr\theta_2 q_2\frac{dv}{dy} + \left(\frac{3}{2}\theta_5 - \theta_2 - \theta_4\frac{\omega}{2}\right)\sigma_{12}\frac{d\theta}{dy} = -\frac{P_0 - \sigma_{22}}{\mu}q_1 \; , \tag{15}$$

$$\left(\frac{3}{2}\theta_5 - \theta_2 - \frac{\theta_4}{2}\omega - \frac{5}{2}\frac{1}{\Pr}\right)\sigma_{22}\frac{d\theta}{dy} + \left(\frac{\theta_4}{2} - \frac{\theta_3}{2}\frac{\sigma_{22}}{(P_0 - \sigma_{22})}\right)\theta\frac{d\sigma_{22}}{dy} + \frac{5}{2}\frac{1}{\Pr}P_0\frac{d\theta}{dy} = -\frac{P_0 - \sigma_{22}}{\mu}q_2. \tag{16}$$

Moreover, one finds

$$\sigma_{11} = -\frac{\mu}{p}\frac{\varpi_6}{8}\sigma_{12}\frac{dv}{dy} - \frac{1}{2}\sigma_{22} \; ; \tag{17}$$

this was used to simplify the other equations.

3.2 Orders of magnitude

In a Couette flow the gradients of velocity and temperature are prescribed, while stress and heat flux result from the response of the gas to the given boundary conditions for temperature and velocity. To find the order of magnitude of the components of stress and heat flux for given values of the gradients and given Knudsen number, the scales of the quantities are made explicit by replacing

$$\mu \to \varepsilon\mu, \quad \sigma_{12} \to \epsilon_{12}\sigma_{12}, \quad \sigma_{22} \to \epsilon_{22}\sigma_{22}, \quad q_1 \to \epsilon_1 q_1, \quad q_2 \to \epsilon_2 q_2. \tag{18}$$

Here, ε is the Knudsen number, and $\epsilon_{12},\ \epsilon_{22},\ \epsilon_1,\ \epsilon_2$ are unknown scaling factors that are to be determined from the equations. The idea is that the scaling factors carry the information about the size of the variables, which appear to be dimensionless. At the end of the treatment, the scaling factors are set to unity, which restores the dimension of the variables.

From the original derivation, or from the Chapman–Enskog expansion, we know that these scaling factors are at best of the order of the Knudsen number, but they might be smaller. Thus, they are of order $\mathcal{O}(\varepsilon^a)$ with $a \geq 1$. The hydrodynamic variables and their gradients are not scaled.

The rescaled equation for σ_{12} reads

$$\begin{aligned} &\epsilon_{22}\left(\frac{\varpi_6}{16}+\frac{\varpi_2}{4}-1\right)\sigma_{22}\frac{dv}{dy} - \varepsilon\epsilon_{12}\frac{\mu}{p}\left(\frac{\varpi_6}{8}-\frac{\varpi_2}{2}\right)\sigma_{12}\frac{dv}{dy}\frac{dv}{dy} + \epsilon_1\frac{1}{5}\Pr\left(\varpi_5-\varpi_3\omega\right)q_1\frac{d\ln\theta}{dy} \\ &-\epsilon_{22}\epsilon_1\frac{\varpi_4}{5}\Pr\frac{q_1}{P_0-\epsilon_{22}\sigma_{22}}\frac{d\sigma_{22}}{dy} + \epsilon_1\frac{\varpi_3}{5}\Pr\frac{dq_1}{dy} + P_0\frac{\partial v}{\partial y} = -\frac{\epsilon_{12}}{\varepsilon}\frac{P_0-\epsilon_{22}\sigma_{22}}{\mu}\sigma_{12}\ . \end{aligned} \tag{19}$$

The only term at order $\mathcal{O}\left(\varepsilon^0\right)$ is the term without a scaling parameter, $P_0\frac{\partial v}{\partial y}$, on the left hand side. Since all unknown scaling parameters are at least of order $\mathcal{O}(\varepsilon)$, by the fact that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms* [20], this term must be matched in order of magnitude by the term on the right, i.e.,

$$\frac{\epsilon_{12}}{\varepsilon} = \mathcal{O}\left(\varepsilon^0\right) \quad \text{so that} \quad \epsilon_{12} = \varepsilon. \tag{20}$$

After scaling, the equation for q_2 reads

$$\begin{aligned} &\epsilon_{22}\left(\frac{3}{2}\theta_5-\theta_2-\frac{\theta_4}{2}\omega-\frac{5}{2}\frac{1}{\Pr}\right)\sigma_{22}\frac{d\theta}{dy} + \epsilon_{22}\left(\frac{\theta_4}{2}-\frac{\theta_3}{2}\frac{\epsilon_{22}\sigma_{22}}{(P_0-\epsilon_{22}\sigma_{22})}\right)\theta\frac{d\sigma_{22}}{dy} \\ &+\frac{5}{2}\frac{1}{\Pr}P_0\frac{d\theta}{dy} = -\frac{\epsilon_2}{\varepsilon}\frac{(P_0-\epsilon_{22}\sigma_{22})}{\mu}q_2\ . \end{aligned} \tag{21}$$

Again, only the last term on the l.h.s. is $\mathcal{O}\left(\varepsilon^0\right)$, the term on the right must be of the same order, that is

$$\frac{\epsilon_2}{\varepsilon} = \mathcal{O}\left(\varepsilon^0\right) \quad \text{so that} \quad \epsilon_2 = \varepsilon. \tag{22}$$

With the previous results for ϵ_{12} and ϵ_2, the scaled equations for σ_{22} and q_1 read

$$\begin{aligned} &\varepsilon\frac{4}{15}\Pr\left(\varpi_5-\varpi_3\omega\right)q_2\frac{d\ln\theta}{dy} - \varepsilon\epsilon_{22}\frac{4}{15}\Pr\varpi_4\frac{q_2}{P_0-\epsilon_{22}\sigma_{22}}\frac{d\sigma_{22}}{dy} \\ &+\varepsilon\left(\frac{\varpi_6}{12}-\frac{2}{3}\varpi_2-\frac{4}{15}\Pr\varpi_3\right)\sigma_{12}\frac{dv}{dy} = -\frac{\epsilon_{22}}{\varepsilon}\frac{P_0-\epsilon_{22}\sigma_{22}}{\mu}\sigma_{22}\ , \end{aligned} \tag{23}$$

$$-\varepsilon\epsilon_{22}\frac{\theta_3}{2}\frac{\theta\sigma_{12}}{p}\frac{d\sigma_{22}}{dy} + \varepsilon\frac{2}{5}\Pr\theta_2 q_2\frac{dv}{dy} + \varepsilon\left(\frac{3}{2}\theta_5-\theta_2-\theta_4\frac{\omega}{2}\right)\sigma_{12}\frac{d\theta}{dy} = -\frac{\epsilon_1}{\varepsilon}\frac{P_0-\epsilon_{22}\sigma_{22}}{\mu}q_1\ . \tag{24}$$

All terms on the left hand sides are at least of order $\mathcal{O}(\varepsilon)$; hence, it follows

$$\epsilon_{22} = \varepsilon^2 \quad \text{and} \quad \epsilon_1 = \varepsilon^2. \tag{25}$$

3.3 Properly scaled equations

The above arguments revealed the proper order of magnitude of all terms in the equations in powers of the Knudsen number ε; the properly scaled equations are

$$\varepsilon^2\left(\frac{\varpi_6}{16}+\frac{\varpi_2}{4}-1\right)\sigma_{22}\frac{dv}{dy}-\varepsilon^2\frac{\mu}{p}\left(\frac{\varpi_6}{8}-\frac{\varpi_2}{2}\right)\sigma_{12}\frac{dv}{dy}\frac{dv}{dy}+\varepsilon^2\frac{1}{5}\,\mathrm{Pr}\,(\varpi_5-\varpi_3\omega)\,q_1\frac{d\ln\theta}{dy}$$
$$-\varepsilon^4\frac{\varpi_4}{5}\,\mathrm{Pr}\,\frac{q_1}{P_0-\varepsilon^2\sigma_{22}}\frac{d\sigma_{22}}{dy}+\varepsilon^2\frac{\varpi_3}{5}\,\mathrm{Pr}\,\frac{dq_1}{dy}+P_0\frac{\partial v}{\partial y}=-\frac{P_0-\varepsilon^2\sigma_{22}}{\mu}\sigma_{12}\,, \tag{26}$$

$$\varepsilon\frac{4}{15}\,\mathrm{Pr}\,(\varpi_5-\varpi_3\omega)\,q_2\frac{d\ln\theta}{dy}-\varepsilon^3\frac{4}{15}\,\mathrm{Pr}\,\varpi_4\frac{q_2}{P_0-\varepsilon^2\sigma_{22}}\frac{d\sigma_{22}}{dy}$$
$$+\varepsilon\left(\frac{\varpi_6}{12}-\frac{2}{3}\varpi_2-\frac{4}{15}\,\mathrm{Pr}\,\varpi_3\right)\sigma_{12}\frac{dv}{dy}=-\varepsilon\frac{P_0-\varepsilon^2\sigma_{22}}{\mu}\sigma_{22}\,, \tag{27}$$

$$-\varepsilon^3\frac{\theta_3}{2}\frac{\theta\sigma_{12}}{p}\frac{d\sigma_{22}}{dy}+\varepsilon\frac{2}{5}\,\mathrm{Pr}\,\theta_2q_2\frac{dv}{dy}+\varepsilon\left(\frac{3}{2}\theta_5-\theta_2-\theta_4\frac{\omega}{2}\right)\sigma_{12}\frac{d\theta}{dy}=-\varepsilon\frac{P_0-\varepsilon^2\sigma_{22}}{\mu}q_1\,. \tag{28}$$

$$\varepsilon^2\left(\frac{3}{2}\theta_5-\theta_2-\frac{\theta_4}{2}\omega-\frac{5}{2}\frac{1}{\mathrm{Pr}}\right)\sigma_{22}\frac{d\theta}{dy}+\varepsilon^2\left(\frac{\theta_4}{2}-\frac{\theta_3}{2}\frac{\varepsilon^2\sigma_{22}}{\left(P_0-\varepsilon^2\sigma_{22}\right)}\right)\theta\frac{d\sigma_{22}}{dy}$$
$$+\frac{5}{2}\frac{1}{\mathrm{Pr}}P_0\frac{d\theta}{dy}=-\frac{P_0-\varepsilon^2\sigma_{22}}{\mu}q_2. \tag{29}$$

3.4 Bulk equations

The first-order equations are obtained by removing all terms that carry a factor ε^a ($a\geq 1$). As one would expect, this gives the NSF equations for Couette flow,

$$\sigma_{12}=-\mu\frac{\partial v}{\partial y},\quad q_2=-\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{d\theta}{dy},\quad \sigma_{22}=0,\quad q_1=0\,. \tag{30}$$

The second-order equations are obtained by adding terms that carry the factor ε^1 and ignoring those with higher powers in ε. This gives again the NSF laws for σ_{12} and q_2 and the leading contributions for σ_{22} and q_1,

$$\sigma_{12}=-\mu\frac{\partial v}{\partial y},\quad \sigma_{22}=-\left(\frac{2}{3}\varpi_2+\frac{4}{15}\,\mathrm{Pr}\,\varpi_3-\frac{\varpi_6}{12}\right)\frac{\sigma_{12}\sigma_{12}}{P_0}-\frac{8}{75}\mathrm{Pr}^2\,(\varpi_3\omega-\varpi_5)\,\frac{q_2q_2}{P_0\theta}, \tag{31}$$
$$q_2=-\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{d\theta}{dy},\quad q_1=\mathrm{Pr}\left(\frac{3}{5}\theta_5-\frac{\omega}{5}\theta_4\right)\frac{\sigma_{12}q_2}{P_0}.$$

For Maxwell molecules this gives the set of equations first presented in [21], see also [18],

$$\sigma_{12}=-\mu\frac{\partial v}{\partial y},\quad q_2=-\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{\partial\theta}{\partial y},\quad \sigma_{22}=-\frac{6}{5}\frac{\sigma_{12}\sigma_{12}}{P_0},\quad q_1=\frac{7}{2}\frac{q_2\sigma_{12}}{P_0}. \tag{32}$$

For hard spheres we obtain

$$\sigma_{22}=-1.1596\frac{\sigma_{12}\sigma_{12}}{P_0}-0.046139\frac{q_2q_2}{\theta P_0},\quad q_1=3.126\frac{q_2\sigma_{12}}{P_0}. \tag{33}$$

Higher order approximations can be obtained in the same manner, but these will not be considered here.

The equations (31) will be termed as "bulk equations". Indeed, Grad's 13 moment equations, and their generalization (7,8), cannot describe Knudsen layers [18,19], and thus the equations are valid only outside the Knudsen layer.

Normal stress, σ_{22}, and the heat flux parallel to the wall, q_1, both vanish in the NSF theory, and thus their non-zero values as given by (31) describe pure rarefaction effects. In particular it must be noted that there is no temperature gradient in the x-direction: q_1 is a heat flux that is not driven by a temperature gradient.

It is worthwhile to note that the bulk equations can be obtained also from the Burnett equations by means of a similar scaling procedure, see the appendix for details.

The solution of the bulk equations for a Couette flow (2,31) requires jump and slip boundary conditions of second order that can be computed from Grad's 13 moment distribution [14,18],

$$V = v^\alpha - v_W^\alpha = \frac{-\frac{2-\chi}{\chi}\alpha_1\sqrt{\frac{\pi}{2}}\sqrt{\theta}\sigma_{12}n^\alpha - \frac{1}{5}\alpha_2 q_1}{\rho\theta + \frac{1}{2}\sigma_{22}}, \quad \theta^\alpha - \theta_W^\alpha = -\frac{\frac{2-\chi}{2\chi}\beta_1\sqrt{\frac{\pi}{2}}\sqrt{\theta}q_2 n^\alpha + \frac{1}{4}\theta\sigma_{22}}{\rho\theta + \frac{1}{2}\sigma_{22}} + \frac{V^2}{4}, \tag{34}$$

with correction factors $\alpha_1 = 1.114$, $\alpha_2 = 1.34533$, $\beta_1 = 1.127$; V is the slip velocity [3,18]. The pressure constant P_0 follows from the prescribed mass between the plates.

Note that the solution of the full 13 moment equations would require an additional boundary condition for σ_{22}. Marquez and Kremer present an analytical solution of Grad's 13 moment equations for a Couette flow under the assumptions of constant pressure and temperature independent viscosity [10]; for hard sphere molecules they find $q_1 = 3.55\frac{q_2\sigma_{12}}{P_0}$.

4 Superpositions of bulk solutions and Knudsen layers

For constructing the final solution, it is assumed that the non-equilibrium quantities can be split into the bulk (B) and Knudsen layer (L) contributions, $\phi = \phi_B + \phi_L$, where the Knudsen layer contributions vanish in some distance from the wall. The bulk contributions will be computed from the equations of the previous section.

4.1 Knudsen layers with the R13 equations

The Knudsen layer contribution is computed from linearizing the R13 equations (2–6) in Couette geometry. Together with the linearized conservation laws, the relevant equations can be reduced to (see Refs. [18,19] for details)

$$p_0\frac{dv}{dy} + \frac{2}{5}\frac{dq_1}{dy} = -\frac{p_0}{\mu_0}\sigma_{12} = \text{const.}, \quad q_1 = \frac{9}{5}\left(\frac{\mu_0\sqrt{\theta_0}}{p_0}\right)^2\frac{d^2q_1}{dy^2},$$

$$\frac{5}{2}\frac{d\theta}{dy} + \frac{\theta_0}{p_0}\frac{d\sigma_{22}}{dy} = -\frac{2}{3}\frac{q_2}{\mu_0} = \text{const.}, \quad \sigma_{22} = \frac{6}{5}\left(\frac{\mu_0\sqrt{\theta_0}}{p_0}\right)^2\frac{d^2\sigma_{22}}{dy^2}. \tag{35}$$

Integration of Eq. (35) gives for velocity and temperature

$$v = v_0 - \frac{\sigma_{12}}{\mu_0}\left(y - \frac{L}{2}\right) - \frac{2}{5}\frac{q_1}{p_0} \quad \text{with} \quad q_1 = A\sinh\left[\sqrt{\frac{5}{9}}\frac{\frac{y}{L}-\frac{1}{2}}{\text{Kn}}\right] + B\cosh\left[\sqrt{\frac{5}{9}}\frac{\frac{y}{L}-\frac{1}{2}}{\text{Kn}}\right], \tag{36}$$

$$\theta = \theta_0 - \frac{4}{15}\frac{q_2}{\mu_0}\left(y - \frac{L}{2}\right) - \frac{2}{5}\frac{\theta_0\sigma_{22}}{p_0} \quad \text{with} \quad \sigma_{22} = C\sinh\left[\sqrt{\frac{5}{6}}\frac{\frac{y}{L}-\frac{1}{2}}{\text{Kn}}\right] + D\cosh\left[\sqrt{\frac{5}{6}}\frac{\frac{y}{L}-\frac{1}{2}}{\text{Kn}}\right]. \tag{37}$$

v_0, σ_{12}, A, B and θ_0, q_1, C, D are constants of integration that must be obtained from boundary conditions. $\text{Kn} = \frac{\mu_0\sqrt{\theta_0}}{p_0 L}$ is the Knudsen number. We can identify $\left(-\frac{2}{5}\frac{q_1}{p_0}\right)$ and $\left(-\frac{2}{5}\frac{\theta_0\sigma_{22}}{p_0}\right)$ as the Knudsen boundary layers for velocity and temperature, according to the R13 equations.

Figure 1 shows the shape of the hyperbolic sine and cosine functions that make up the Knudsen layers. It can be seen that for $\text{Kn} \leq 0.05$ the Knudsen layers vanish in the bulk (that is in some distance away from the wall). For $\text{Kn} \geq 0.1$, however, the Knudsen layers contribute through the whole domain.

Grad's 13 moment equations and the generalized 13 moment equations do not give linear Knudsen layers. The Burnett equations yield Knudsen layers for density ρ and σ_{22}, but not for v, θ and q_1. The super-Burnett equations give periodic solutions of the form $A\sin\left[\lambda\frac{x-1/2}{\text{Kn}}\right]$, $B\cos\left[\lambda\frac{x-1/2}{\text{Kn}}\right]$, which are unphysical [18].

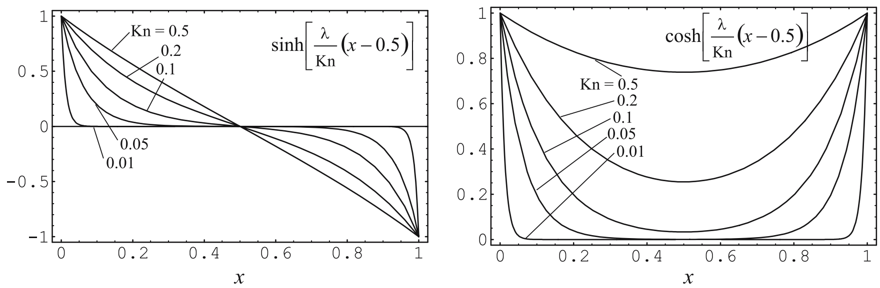

Fig. 1 Knudsen layer contributions for Knudsen numbers between 0.01 and 0.5

4.2 Superpositions

The superposition of bulk solution and Knudsen layers for the R13 equations gives

$$v = v_{|B} - \frac{2}{5}\frac{q_{1|L}}{P_0}, \quad \theta = \theta_{|B} - \frac{2}{5}\frac{\theta_0 \sigma_{22|L}}{P_0}, \quad \sigma_{12} = \sigma_{12|B}, \quad \sigma_{22} = \sigma_{22|B} + \sigma_{22|L},$$
$$p = P_0 - \sigma_{22|L}, \quad \rho = \frac{p}{\theta}, \quad q_1 = q_{1|B} + q_{1|L}, \quad q_2 = q_{2|B}. \tag{38}$$

The constants of integration, A, B, C, D, should be computed from additional boundary conditions for normal stress, σ_{22}, and parallel heat flux, q_1. Since at present no set of reliable boundary conditions is available, the constants were obtained by fitting to DSMC simulations for Maxwell molecules. The coefficients A, D for the simulations are

	Kn = 0.05	Kn = 0.1	Kn = 0.5
$v_W^L = 200\frac{\mathrm{m}}{\mathrm{s}}$	$A = 0.009$ $D = 0.0015$	$A = 0.015$ $D = 0.003$	$A = 0.03$ $D = 0.02$

;

due to the symmetry of the results, one finds $B = C = 0$.

The figures compare results of DSMC calculations, NSF equations with first-order jump and slip boundary conditions, and the R13 equations (superpositions) for $v_W^0 = 0$, $v_W^L = 200\frac{m}{s}$, $\theta_W^0 = \theta_W^L = 273K$ and Knudsen numbers Kn $= 0.05$ (Fig. 2), for Kn $= 0.1$ (Fig. 3), and Kn $= 0.5$ (Fig. 4). The numerical method for solving the bulk equations with jump and slip boundary conditions is presented in [21].

For Kn $= 0.05$ and Kn $= 0.1$ the superposition matches the DSMC simulations quite well; the most visible differences lie in the bulk values for σ_{12} and σ_{22}. Due to the quadratic term $\frac{V^2}{4}$ in the jump condition $(34)_2$, the temperature maximum is reproduced very well, while some differences can be observed at the boundaries.

The NSF, on the other hand, can neither describe Knudsen boundary layers nor the rarefaction effects described by the bulk contributions to σ_{22} and q_1. Nevertheless, the NSF results for the hydrodynamic variables ρ, θ, v and the classical fluxes σ_{12} and q_2 agree quite well with the DSMC simulations. The NSF temperature curve lies too low, inclusion of the term $\frac{V^2}{4}$ into the temperature boundary condition would improve the results further.

We note that for Kn $= 0.05$ the Knudsen layers do not contribute to the result in the middle of the domain (see Fig. 1). Thus the result in the middle of the domain is governed by the bulk solution which gives excellent agreement to the DSMC results. Moreover, for Kn $= 0.05$ and Kn $= 0.1$, the computed shape of the Knudsen layers agrees perfectly with the DSMC simulations.

For Kn $= 0.5$ we notice marked deviations in the curves for the hydrodynamic variables and for shear stress σ_{12} and normal heat flux q_2. These are due to a lack of accuracy of the bulk equations, which are only of second order in Kn. It can be expected that consideration of higher order terms, i.e., of the full non-linear R13 equations, which are of third-order accuracy in Kn, will give a better match.

In Ref. [10], Marquez and Kremer present an analytical solution of Grad's 13 moment equations for Couette flow under the assumption of constant pressure. Their solution agrees with our bulk solution, but does not include boundary layer contributions which are not contained in the Grad 13 equations, but arise only

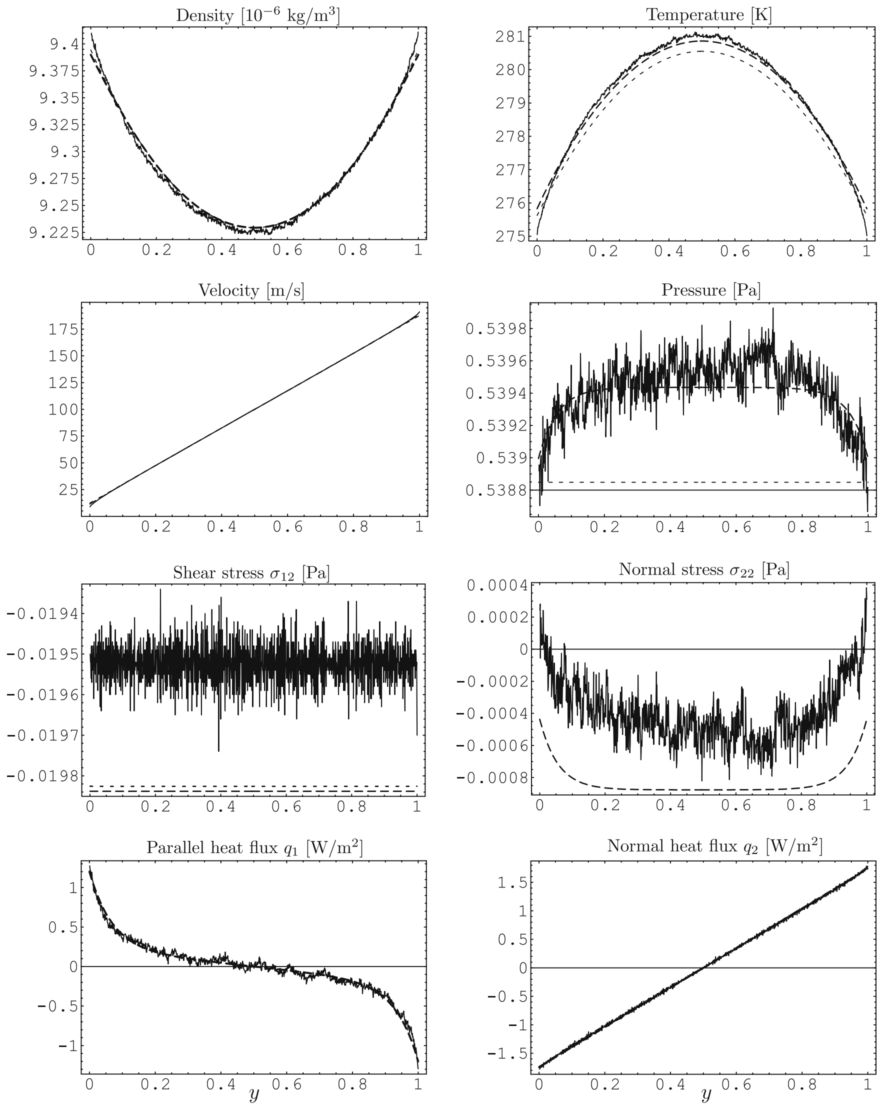

Fig. 2 Couette flow at Kn $= 0.05$,with $v_W^L = 200\,\frac{\mathrm{m}}{\mathrm{s}}$. *Continuous line*: DSMC, *finely dashed line*: NSF, *dashed line*: superposition of bulk solution and linear Knudsen layer solution. Recall that NSF implies $q_1 = \sigma_{22} = 0$ (curves not shown)

in higher order theories. The solution of the bulk equations allows the direct control of the boundary values for temperature and velocity, while the analytical solution in Ref. [10] needs the shear rate as input, and then deduces the corresponding boundary values for temperature and velocity.

It is worthwhile to note that only the superpositions of linear Knudsen layers and bulk solution gives good agreement over the whole range of velocities. The non-linear contributions of the bulk solution can be ignored only at very low velocities (far lower than those presented) [19]. For the cases presented here, both contributions are equally important. At larger velocities the non-linear contributions play a more marked role. For those cases the solution of Ref. [10] agrees better with DSMC.

Due to the fitting of the Knudsen layer amplitudes, the presented results are not self-contained. We are currently working on developing additional boundary conditions for q_1 and σ_{22} and hope that we will be able to present a set of reliable additional boundary conditions in the future.

Obviously, the presented superposition of bulk equations and Knudsen layers can only be done in simple geometries, e.g., Couette and Poiseuille flows. We see the present results as a test for the reliability and as proof for the richness of the basic equations solved, the R13 equations.

The NSF equations can describe the basic hydrodynamic features of the flows reasonably well, but cannot describe any rarefaction effects other than jump and slip. For Couette flow, hydrodynamic (ρ, θ, v, σ_{12}, q_2) and

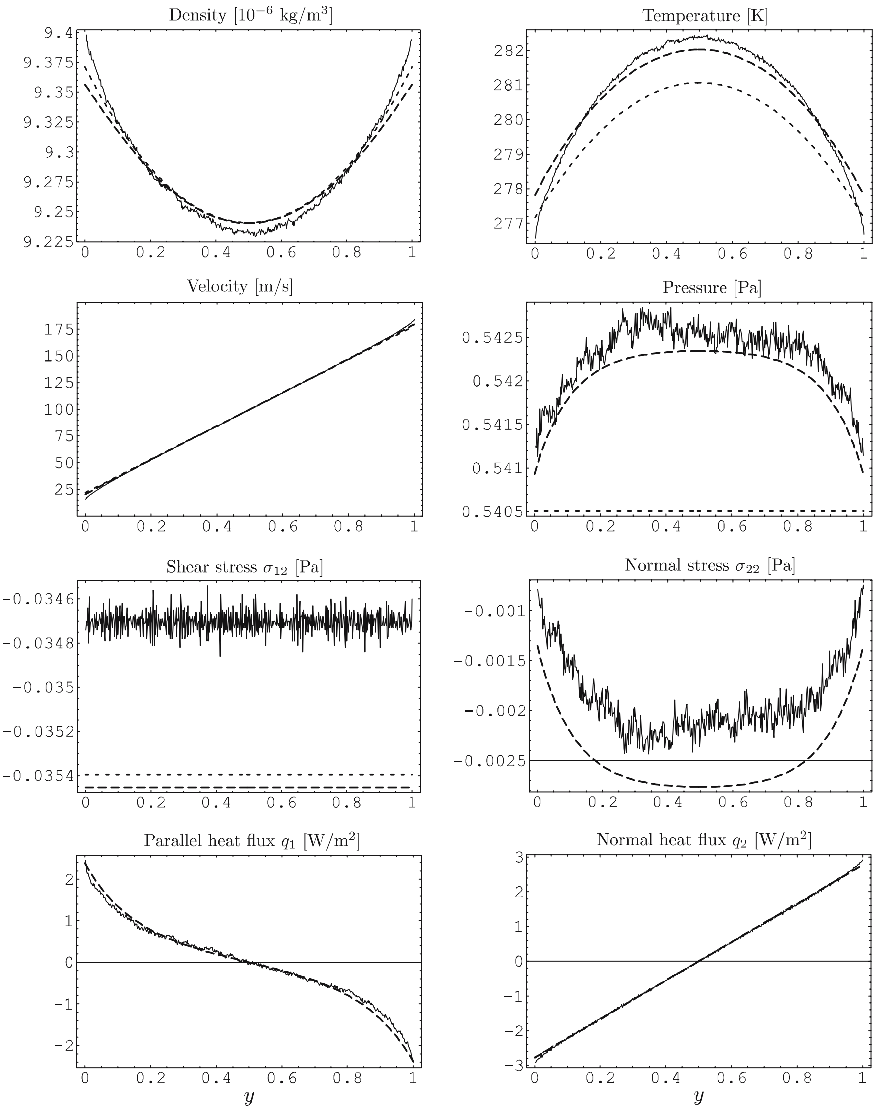

Fig. 3 Couette flow at Kn = 0.1,with $v_W^L = 200\,\frac{\mathrm{m}}{\mathrm{s}}$. *Continuous line*: DSMC, *finely dashed line*: NSF, *dashed line*: superposition of bulk solution and linear Knudsen layer solution. Recall that NSF implies $q_1 = \sigma_{22} = 0$ (curves not shown)

rarefaction (σ_{22}, q_1) effects are geometrically decoupled. For more complex flow, one will expect a stronger coupling, which might lead to stronger deviations. In any case, the non-linear term $\frac{V^2}{4}$ should be included in the temperature jump boundary conditions for the NSF equations.

It should be noted that, for the presented results, the computational time is comparable to solution of the NSF equations (few seconds), while the computation of the DSMC results takes hours or days depending on Knudsen number and velocity (small Knudsen numbers require many simulated particles, computations with small velocities have strong numerical noise, in both cases run times must be large).

In conclusion we state that the R13 equations gives reliable results for boundary value problems, including the description of Knudsen boundary layers and rarefaction effects, like heat flux without a temperature gradient. The R13 equations and their reduction to the bulk equations are derived by means of the order of magnitude method. Due to the success of the equations the order of magnitude method must be considered as a useful and reliable tool for the reduction of models.

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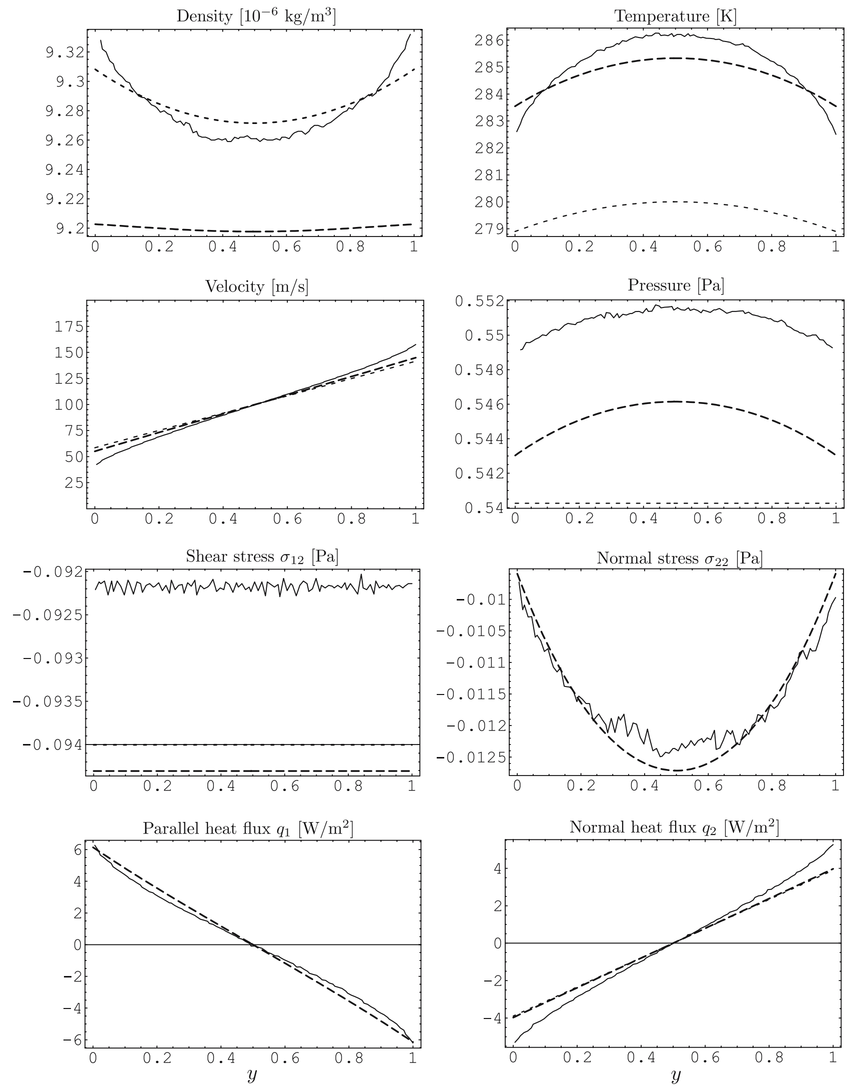

Fig. 4 Couette flow at $\mathrm{Kn} = 0.5$,with $v_W^L = 200\,\frac{\mathrm{m}}{\mathrm{s}}$. *Continuous line*: DSMC, *finely dashed line*: NSF, *dashed line*: superposition of bulk solution and linear Knudsen layer solution. Recall that NSF implies $q_1 = \sigma_{22} = 0$ (curves not shown)

Appendix: Reduction of Burnett equations

In Couette geometry, the Burnett equations [8,18] reduce to

$$\sigma_{12}^{(B)} = -\mu \frac{dv}{dy}, \quad q_2^{(B)} = -\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{dT}{dy},$$

$$\sigma_{22}^{(B)} = -\frac{\mu^2}{p}\frac{2}{3}\left[\varpi_2 \frac{d}{dy}\left(\frac{\theta}{p}\frac{dp}{dy}\right) - \varpi_3 \frac{d^2\theta}{dy^2} - \varpi_4 \frac{d\theta}{dy}\frac{d\ln p}{dy} + \frac{\varpi_5}{\theta}\frac{d\theta}{dy}\frac{d\theta}{dy} + \left(\varpi_2 - \frac{\varpi_6}{8}\right)\frac{dv}{dy}\frac{dv}{dy}\right] \tag{39}$$

$$q_1^{(B)} = \frac{\mu^2}{p}\left[\frac{3}{2}\theta_5 \frac{dv}{dy}\frac{d\theta}{dy} + \frac{\theta_3}{2}\frac{dv}{dy}\frac{\theta}{p}\frac{dp}{dy} + \frac{\theta_4}{2}\theta\frac{d^2v}{dy^2}\right]$$

Insertion of $\sigma_{12}^{(B)}, q_2^{(B)}$ into the conservation laws $(12)_{1,3}$ shows, with $d\ln\mu = d\ln\theta^{\omega} = \omega d\ln\theta$, that

$$\frac{d^2v}{dy^2} = -\frac{d\ln\mu}{dy}\frac{dv}{dy} = -\frac{\omega}{\theta}\frac{d\theta}{dy}\frac{dv}{dy}, \quad \frac{d^2\theta}{dy^2} = -\frac{2}{5}\,\mathrm{Pr}\,\frac{dv}{dy}\frac{dv}{dy} - \frac{\omega}{\theta}\frac{d\theta}{dy}\frac{d\theta}{dy}. \tag{40}$$

With this, the equations for $\sigma_{22}^{(B)}$, $q_1^{(B)}$ become

$$\sigma_{22}^{(B)} = -\frac{\mu^2}{p}\left[\left(\frac{4}{15}\varpi_3 \operatorname{Pr} - \frac{\varpi_6}{12} + \frac{2}{3}\varpi_2\right)\frac{dv}{dy}\frac{dv}{dy} + \frac{2}{3}\left(\varpi_3\omega - \varpi_5\right)\frac{1}{\theta}\frac{d\theta}{dy}\frac{d\theta}{dy}\right] + \underline{\frac{\mu^2}{p}\frac{2}{3}\left[\left(\varpi_4 - \varpi_2\right)\frac{d\theta}{dy}\frac{d\ln p}{dy} - \varpi_2\theta\frac{d^2\ln p}{dy^2}\right]} \tag{41}$$

$$q_1^{(B)} = -\frac{\mu^2}{p}\left(\frac{\omega}{2}\theta_4 - \frac{3}{2}\theta_5\right)\frac{dv}{dy}\frac{d\theta}{dy} + \underline{\frac{\theta_3}{2}\frac{\mu^2\theta}{p^2}\frac{dv}{dy}\frac{dp}{dy}} \tag{42}$$

at first glance, the underlined terms are of second order (as can be seen form the factor $\mu^2 \propto \varepsilon^2$). However, the conservation of normal momentum gives $dp = -d\sigma_{22}$ and σ_{22} is of second order. Thus, the terms involving pressure derivatives lead to terms of fourth order in ε. Since the Burnett equations were derived by considering second-order terms in ε, these fourth-order terms must be ignored. Note that a proper theory of the fourth order would contain additional terms of the fourth order, so that, obviously, the Burnett equations are not correct to fourth order. As was shown in [18], these terms would lead to Knudsen layers for density, and σ_{22}. Removal of the underlined terms recovers the bulk equations (31) when gradients of velocity and temperature are replaced by σ_{12} and q_2.

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