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Original article

Burnett equations for the ellipsoidal statistical BGK model

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Abstract. In order to discuss the agreement of the ellipsoidal statistical BGK (ES-BGK) model with the Boltzmann equation, Burnett equations are computed by means of the second-order Chapman–Enskog expansion of the ES-BGK model. It is found that the Burnett equations for the ES-BGK model with the correct Prandtl number are identical to the Burnett equations for the Boltzmann equation for Maxwell molecules (fifth-order power potentials). However, for other types of particle interaction, the Boltzmann Burnett equations cannot be reproduced from the ES-BGK model.
Furthermore, the linear stability of the ES-BGK Burnett equations is discussed. It is shown that the ES-BGK Burnett equations are linearly stable for Prandtl numbers of $1 \leq \mathrm{Pr} \leq 5/4$ and for $\mathrm{Pr} \to \infty$, while they are linearly unstable for $2/3 \leq \mathrm{Pr} < 1$ and $5/4 < \mathrm{Pr} < \infty$.

Key words: Kinetic theory, ES-BGK model Chapman-Enskog expansion, Burnett equations
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1. Introduction

The Boltzmann equation is the fundamental equation of rarefied gas dynamics, and macroscopic constitutive equations can be obtained from the Boltzmann equation through the Chapman–Enskog expansion [1–3]. The Euler equations follow from the zeroth-order expansion, the Navier–Stokes-Fourier equations are obtained from the first-order expansion, and the second-order expansion results in the Burnett equations.

The Boltzmann equation is a nonlinear integro-differential equation, and is difficult to handle. Therefore, some alternative, simpler expressions have been proposed to replace the Boltzmann collision term. These are known as collision models, and any Boltzmann-like equation in which the Boltzmann collision integral is replaced by a collision model is called a model equation or a kinetic model [4].

Due to its simplicity compared with the Boltzmann equation, the BGK equation is widely used in the kinetic theory of gases [5, 6]. While the BGK equation gives qualitatively good results, it fails when one is interested in quantitatively correct results. This fact manifests itself most prominently in the computation of the Prandtl number Pr. While measurements and the theory of the full Boltzmann equation give $\mathrm{Pr} = 2/3$, one obtains $\mathrm{Pr} = 1$ from the standard BGK model [4].

The Gaussian BGK model, or ellipsoidal statistical model (ES-BGK model) [7, 8] is a modification of the standard BGK equation that gives the proper Prandtl number $\mathrm{Pr} = 2/3$. Only recently, Andries et al. succeeded in proving the validity of the H-Theorem for the ES-BGK model [9], which has revived the interest in this model.

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Convenient numerical algorithms are available for BGK-type equations, which allow fast and accurate solutions [10, 11], while direct numerical solutions of the Boltzmann equation are more costly than solutions of the ES-BGK equation, and the Direct Simulation Monte Carlo (DSMC) approach suffers from noise in some transport regimes. Thus, the ES-BGK model might be useful as a substitute for the Boltzmann equation in simulations of rarefied gas flows. Of course, the Prandtl number alone is not a sufficient measure of the agreement between the Boltzmann and ES-BGK equations, all the more so since the Prandtl number is independent of rarefaction effects. One way to study the agreement and differences between the two transport equations is to compare their corresponding Burnett equations. Since the Burnett equations follow from the second-order Chapman–Enskog expansion in the Knudsen number, they include some rarefaction effects. The computation of the Burnett equations for the ES-BGK model, and the comparison with the corresponding equations as obtained from the full Boltzmann equation forms the first part of this paper.

It is well known that the Burnett equations for Maxwell molecules are linearly unstable [12], and thus cannot be used in numerical simulations. Surprisingly, the Burnett equations of the standard BGK model ($\mathrm{Pr} = 1$) [13, 14] are linearly stable [15, 16], which indicates that the stability of the Burnett equations is related to the Prandtl number. The ES-BGK model allows the Prandtl number to be varied in the range $2/3 < \mathrm{Pr} < \infty$, and thus allows the study of this relation. The second part of this paper therefore considers the linear stability of the ES-BGK Burnett equations for varying Prandtl number. It will be shown that the equations are linearly stable for $1 \leq \mathrm{Pr} \leq 5/4$ and for $\mathrm{Pr} \to \infty$, while they are linearly unstable for $2/3 \leq \mathrm{Pr} < 1$ and $5/4 < \mathrm{Pr} < \infty$. This result implies that the instability of the Burnett equations is not inherent to the Chapman–Enskog method. It must be noted, however, that the second-order Chapman–Enskog expansion leads to unstable equations in most cases [17].

The remainder of the paper is organized as follows: In Sect. 2 we reintroduce the ES-BGK equation and discuss its Chapman–Enskog expansion in general. Section 3 presents the first three levels of the expansion, that is, the corresponding Euler, Navier-Stokes-Fourier, and Burnett equations. The latter are compared with the results from the Boltzmann equation. Section 4 deals with the relation between linear stability and Prandtl number. The paper ends with our conclusions.

2. Chapman–Enskog expansion of the ES-BGK model

The ES-BGK model (ellipsoidal statistical BGK), also known as the Gaussian-BGK model [7–9, 18], reads, when external forces are omitted,

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = S, \quad \text{where} \quad S = \nu_G \left(f_G - f \right) \ . \tag{1}$$

Here, $f\left(t,\, x_i,\, c_i\right)$ is the distribution function, $\mathbf{c}$ is the microscopic particle velocity, t is the time, $\mathbf{x}$ is the position, ν_G is the collision frequency, which is a constant with respect to the velocity $\mathbf{C}$ in this model, $\mathbf{C} = \mathbf{c} - \mathbf{u}$ is the peculiar velocity, $\mathbf{u}$ is the macroscopic flow velocity, and S is the collision term. The reference distribution function f_G is a local anisotropic Gaussian,

$$f_G = \frac{\rho}{m} \frac{1}{\sqrt{2\pi \, \det\left[\lambda_{ij}\right]}} \exp\left(-\frac{1}{2} \varepsilon_{ij} C_i C_j \right) \quad , \tag{2}$$

where

$$\lambda_{ij} = (1-b) RT \delta_{ij} + \frac{b}{\rho} p_{ij}, \quad \text{and} \quad \varepsilon_{ij} = \lambda_{ij}^{-1} \ . \tag{3}$$

Here m is the mass of one microscopic particle, ρ is the mass density, T is the temperature, R is the gas constant, and p_{ij} is the pressure tensor, which will be defined below. Moreover, b is a parameter, which serves to adjust the Prandtl number, and must lie in the interval $[-1/2,\, 1]$ to ensure that λ_{ij} is positive definite [7, 9]. δ_{ij} denotes the unit tensor.

When $b = 0$, the Gaussian f_G reduces to the Maxwellian f_m, and the ES-BGK model becomes the standard BGK model [5, 6].

In the macroscopic continuum theory of rarefied gas dynamics, the state of the gas is described by macroscopic parameters, such as the mass density ρ, macroscopic flow velocity $\mathbf{u}$, temperature T, and others, which depend on position **x** and time t. These quantities can be recovered from the distribution f by taking velocity averages (moments) of the corresponding microscopic parameters,

$$\rho = m\int f d\mathbf{c}, \quad \rho u_i = m\int c_i f d\mathbf{c}, \quad \rho e = \frac{3}{2}p = \frac{3}{2}\rho RT = \int \frac{m}{2}C^2 f d\mathbf{c} \,,$$
$$p_{ij} = m\int f C_i C_j dc = p\delta_{ij} + \sigma_{ij} \quad \text{with} \quad \sigma_{ij} = m\int f C_{<i}C_{j>} d\mathbf{c}, \; q_i = \frac{m}{2}\int C^2 C_i f d\mathbf{c} \,, \tag{4}$$

where ρ_e is the density of internal energy, p is the hydrostatic pressure, σ_{ij} is the trace-free part of the pressure tensor, and q_i is the heat flux. The expression $(4)_3$ gives the definition of temperature through the ideal gas law.

A key non-dimensional parameter for describing the rarefaction effect in rarefied gas dynamics is the local Knudsen number $\mathrm{Kn} = l/d$, which is defined as the ratio of the local mean free path of particles, l, over a characteristic length d of the process [19, 20]. The usual continuum equations (e.g. the Navier–Stokes–Fourier equations) are only applicable in the case of small Knudsen numbers, $\mathrm{Kn} << 1$.

Multiplying the Boltzmann equation or kinetic model equations successively by m, mc_i, and $\frac{m}{2}c^2$, then integrating the resulting equations over the microscopic velocity **c**, yields the conservation laws for mass, momentum, and energy,

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \,, \\
&\frac{\partial \rho u_i}{\partial t} + \frac{\partial(\rho u_i u_j + p_{ij})}{\partial x_j} = 0 \,, \\
&\frac{\partial\left(\rho e + \frac{1}{2}\rho u^2\right)}{\partial t} + \frac{\partial\left[\rho e u_j + \frac{1}{2}\rho u^2 u_j + u_i p_{ij} + q_j\right]}{\partial x_j} = 0 \,.
\end{aligned} \tag{5}$$

Here, we have used the conservation conditions for mass, momentum, and energy,

$$m\int S d\mathbf{c} = 0, \quad m\int c_i S d\mathbf{c} = 0, \quad \frac{m}{2}\int c^2 S d\mathbf{c} = 0 \,. \tag{6}$$

Note that the set of conservation laws (5) is not a closed set of equations for the thermodynamic variables ρ, u_i, and T, unless additional equations for σ_{ij} and q_i are given.

For small Knudsen numbers, the Chapman–Enskog method is the traditional way to obtain the constitutive relations for σ_{ij} and q_i. This method will be described briefly here, while the reader is referred to [1–4] for a detailed description.

The Chapman–Enskog method is a perturbation method, which seeks the distribution function in the form of a series in powers of a small dimensionless parameter δ, so that

$$f = \sum_{n=0}^{\infty} \delta^n f^{(n)} \,. \tag{7}$$

Normally δ stands for the Knudsen number Kn. The smallness parameter is also introduced into the transport equation, which now reads

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} = \frac{\nu_G}{\delta}\left(f_G - f\right) \,. \tag{8}$$

Substituting the expression (7) for f into the above equation, and equating the coefficients of the same power of δ yields a set of equations for the expansion coefficients $f^{(n)}$, and its solutions yields the asymptotic solution for the distribution function. The smallness parameter is set to unity ($\delta = 1$) in the ultimate expression.

According to the ordering hypothesis, the trace-free pressure tensor σ_{ij} and heat flux q_i can also be written as series:

$$\begin{aligned}
\sigma_{ij} &= \sum_{n=0}^{\infty} \delta^n \sigma_{ij}^{(n)} = \sum_{n=0}^{\infty} \delta^n \cdot m\int C_{<i}C_{j>} f^{(n)} d\mathbf{c} \,, \\
q_i &= \sum_{n=0}^{\infty} \delta^n q_i^{(n)} = \sum_{n=0}^{\infty} \delta^n \cdot \frac{m}{2}\int C^2 C_i f^{(n)} d\mathbf{c} \,.
\end{aligned} \tag{9}$$

The conserved quantities ρ, u_i, and T, however, are not expanded. That is, in the Chapman–Enskog expansion one assumes that each order of the expansion gives ρ, u_i, and T, which is tantamount to

$$\{\rho, \; \rho u_i, 3p\} = m\int \left\{1, c_i, C^2\right\} f d\mathbf{c} = m\int \left\{1, c_i, \; C^2\right\} f^{(0)} d\mathbf{c} \,,$$

and implies the so-called compatibility conditions

$$m\int\left\{1,\ c_i,\ C^2\right\} f^{(n)}d\mathbf{c}=\{0,\,0,\,0\}\quad\text{for all}\quad n\geq 1\,. \tag{10}$$

Note that, because of $p_{ij}=p\delta_{ij}+\sigma_{ij}$, it follows that $p_{ij}^{(n)}=\sigma_{ij}^{(n)}$ for all $n\geq 1$.

Since the reference distribution function f_G is expressed in terms of the pressure tensor in the ES-BGK model, f_G must also be expressed in the form of powers of δ in the expansion

$$f=\sum_{n=0}^{\infty}\delta^n f^{(n)}\,. \tag{11}$$

This expression for $f_G^{(n)}$ will be derived later in this section.

The time derivative is likewise expressed as a series of operators,

$$\frac{\partial U_r}{\partial t}=\sum_{n=0}^{\infty}\delta^n\frac{\partial_n U_r}{\partial t}\,, \tag{12}$$

where $U_r=\rho,\ u_i,\ T\ (r=1\ldots5)$. Utilizing (5), (9), and (10), one obtains

$$\begin{aligned}
&\frac{\partial_0\rho}{\partial t}=-\frac{\partial\rho\,u_i}{\partial x_i},\quad\frac{\partial_m\rho}{\partial t}=0\,,\\
&\frac{\partial_0 u_i}{\partial t}=-u_j\frac{\partial\,u_i}{\partial x_j}-\frac{1}{\rho}\frac{\partial(p\delta_{ij}+\sigma_{ij}^{(0)})}{\partial x_j},\quad\frac{\partial_m u_i}{\partial t}=-\frac{1}{\rho}\frac{\partial\sigma_{ij}^{(m)}}{\partial x_j}\,,\\
&\frac{\partial_0 T}{\partial t}=-u_i\frac{\partial T}{\partial x_i}-\frac{2}{3\rho\,R}\left(\frac{\partial q_i^{(0)}}{\partial x_i}+\left(p\delta_{ij}+\sigma_{ij}^{(0)}\right)\frac{\partial u_i}{\partial x_j}\right),\quad\frac{\partial_m T}{\partial t}=-\frac{2}{3\rho\,R}\left(\frac{\partial q_i^{(m)}}{\partial x_i}+\sigma_{ij}^{(m)}\frac{\partial u_i}{\partial x_j}\right)\,,
\end{aligned} \tag{13}$$

where $m\geq 1$.

The next assumption of the Chapman–Enskog method is that the distribution function f depends on position x_i and time t only through the hydrodynamic variables U_r and their space derivatives. That is to say, $f^{(n)}=f^{(n)}\left(U_r,\,\nabla U_r,\,...,\,\nabla^n U_r;\,c_i\right)$, but $f^{(n)}$ is not a function of $\nabla^{n+k}U_r$, for $k\geq 1$.

Finally, based on the above line of arguments, the sequence of equations for $f^{(n)}$ can be written as

$$\sum_{n=0}^{\infty}\delta^n\left(\frac{\partial_n f}{\partial t}+c_i\frac{\partial f^{(n)}}{\partial x_i}\right)=\sum_{n=0}^{\infty}\nu_G\delta^{n-1}\left(f_G^{(n)}-f^{(n)}\right)\,, \tag{14}$$

where

$$\begin{aligned}
\frac{\partial_n f}{\partial t}&=\sum_{\substack{m+k=n\\ m,k\geq 0}}\left(\frac{\partial f^{(m)}}{\partial U_r}\cdot\frac{\partial_k U_r}{\partial t}+\frac{\partial f^{(m)}}{\partial\left(\nabla U_r\right)}\cdot\frac{\partial_k\left(\nabla U_r\right)}{\partial t}+...+\frac{\partial f^{(m)}}{\partial\left(\nabla^m U_r\right)}\cdot\frac{\partial_k\left(\nabla^m U_r\right)}{\partial t}\right)\,,\\
\frac{\partial f^{(n)}}{\partial x_i}&=\frac{\partial f^{(n)}}{\partial U_r}\cdot\frac{\partial U_r}{\partial x_i}+\frac{\partial f^{(n)}}{\partial\left(\nabla U_r\right)}\cdot\frac{\partial\left(\nabla U_r\right)}{\partial x_i}+...+\frac{\partial f^{(n)}}{\partial\left(\nabla^n U_r\right)}\cdot\frac{\partial\left(\nabla^n U_r\right)}{\partial x_i}\,,
\end{aligned}$$

with

$$\frac{\partial_k\left(\nabla^m U_r\right)}{\partial t}=\nabla^m\left(\frac{\partial_k U_r}{\partial t}\right)\,.$$

In order to obtain the expression for $f^{(2)}$ and the Burnett equations, we need to know $f_G^{(n)}$ up to the second order in δ. From the definition of the matrix λ_{ij} in (3) one obtains

$$\begin{aligned}
\lambda_{ij}&=RT\delta_{ij}+\frac{b}{\rho}\sigma_{ij}^{(1)}\delta+\frac{b}{\rho}\sigma_{ij}^{(2)}\delta^2+O\left(\delta^3\right)\,,\\
|\lambda|&=(RT)^3\left(1-\left(\frac{b}{p}\right)^2\frac{\sigma_{kn}^{(1)}\sigma_{kn}^{(1)}}{2}\delta^2\right)+O\left(\delta^3\right)\,,\\
\varepsilon_{ij}&=\lambda_{ij}^{-1}=\frac{\delta_{ij}}{RT}-\frac{b}{pRT}\sigma_{ij}^{(1)}\delta+\frac{b}{pRT}\left(\frac{b}{p}\sigma_{ik}^{(1)}\sigma_{kj}^{(1)}-\sigma_{ij}^{(2)}\right)\delta^2+O\left(\delta^3\right)\,.
\end{aligned} \tag{15}$$

It follows for the expansion of the Gaussian that

$$f_G = f_G^{(0)} + f_G^{(1)}\delta + f_G^{(2)}\delta^2 + O\left(\delta^3\right) \ , \tag{16}$$

where

$$\begin{aligned}
f_G^{(0)} &= f_m = \frac{\rho}{m}\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{C^2}{2RT}\right) \ , \\
f_G^{(1)} &= f_m\left(\frac{b}{2pRT}\sigma_{ij}^{(1)}C_iC_j\right) \ , \\
f_G^{(2)} &= f_m\left(\frac{b}{2p}\right)\left[\frac{b}{2p}\sigma_{kn}^{(1)}\sigma_{kn}^{(1)} - \frac{1}{RT}\left(\frac{b}{p}\sigma_{ik}^{(1)}\sigma_{kj}^{(2)} - \sigma_{ij}^{(2)}\right)C_iC_j + \frac{b}{4p(RT)^2}\sigma_{ij}^{(1)}\sigma_{kl}^{(1)}C_iC_jC_kC_l\right] \ .
\end{aligned} \tag{17}$$

Here we have anticipated that $\sigma_{ij}^{(0)} = 0$, as will be shown below. f_m is the local Maxwellian distribution.

3. Burnett equations for the ES-BGK model

3.1. Euler equations

First, let us consider the order of $O(1/\delta)$ in (14), which yields the constitutive relations for the Euler equations,

$$f^{(0)} = f_G^{(0)} = f_m \ , \tag{18}$$

$$p_{ij}^{(0)} = p\delta_{ij}, \quad \sigma_{ij}^{(0)} = 0, \quad q_i^{(0)} = 0 \ . \tag{19}$$

3.2. Navier-Stokes-Fourier equations

Next, we consider the order $O\left(\delta^0\right)$ in (14), which yields

$$f^{(1)} = f_G^{(1)} - \frac{1}{\nu_G}\left(\frac{\partial_0 f}{\partial t} + c_i\frac{\partial f^{(0)}}{\partial x_i}\right) \ ,$$

where

$$\begin{aligned}
\frac{\partial_0 f}{\partial t} &= \frac{\partial f^{(0)}}{\partial U_r}\cdot\frac{\partial_0 U_r}{\partial t} = \frac{\partial f^{(0)}}{\partial \rho}\cdot\frac{\partial_0 \rho}{\partial t} + \frac{\partial f^{(0)}}{\partial u_i}\cdot\frac{\partial_0 u_i}{\partial t} + \frac{\partial f^{(0)}}{\partial T}\cdot\frac{\partial_0 T}{\partial t} \ , \\
\frac{\partial f^{(0)}}{\partial x_i} &= \frac{\partial f^{(0)}}{\partial U_r}\cdot\frac{\partial U_r}{\partial x_i} = \frac{\partial f^{(0)}}{\partial \rho}\cdot\frac{\partial \rho}{\partial x_i} + \frac{\partial f^{(0)}}{\partial u_j}\cdot\frac{\partial u_j}{\partial x_i} + \frac{\partial f^{(0)}}{\partial T}\cdot\frac{\partial T}{\partial x_i} \ .
\end{aligned}$$

After some manipulation, one obtains

$$f^{(1)} = f_m\frac{b}{2pRT}\sigma_{ij}^{(1)}C_iC_j - \frac{f_m}{\nu_G}\left(\frac{C_iC_j}{RT}\cdot\frac{\partial u_{<i}}{\partial x_{j>}} + \frac{C_i}{T}\frac{\partial T}{\partial x_i}\left(\frac{C^2}{2RT} - \frac{5}{2}\right)\right) \ , \tag{20}$$

$$\sigma_{ij}^{(1)} = -2\mu\frac{\partial u_{<i}}{\partial x_{j>}} = -\frac{2}{1-b}\frac{p}{\nu_G}\frac{\partial u_{<i}}{\partial x_{j>}}, \quad q_i^{(1)} = -\kappa\frac{\partial T}{\partial x_i} = -\frac{5}{2}\frac{Rp}{\nu_G}\frac{\partial T}{\partial x_i} \ . \tag{21}$$

The constitutive relations in (21) are the laws of Navier–Stokes and Fourier, where $\mu = \frac{1}{1-b}\frac{p}{\nu_G}$ is the viscosity, and $\kappa = \frac{5}{2}\frac{Rp}{\nu_G}$ is the thermal conductivity. It follows that the Prandtl number is related to the coefficient b of the ES-BGK model by the relation

$$\Pr = \frac{5R\mu}{2\kappa} = \frac{1}{1-b} \ . \tag{22}$$

By means of the Navier–Stokes and Fourier laws (21), the phase density can be rewritten in a form that is familiar from the Chapman–Enskog expansion of the Boltzmann equation of Maxwell molecules, viz.

$$f^{(1)} = -f_m\left(\frac{C_iC_j}{RT}\frac{\mu}{p}\frac{\partial u_{<i}}{\partial x_{j>}} + \left(\frac{C^2}{2RT} - \frac{5}{2}\right)C_i\frac{2}{5}\frac{\kappa}{pRT}\frac{\partial T}{\partial x_i}\right) \ . \tag{23}$$

It is straightforward to show that this expression indeed satisfies the compatibility relations (10).

3.3. Viscosity and collision frequency

For ideal gases, it is well known that the viscosity μ is a function of temperature, and can be written as [2]

$$\mu(T) = \mu_0 \left(\frac{T}{T_0}\right)^\beta , \tag{24}$$

where μ_0 is the viscosity at the reference temperature T_0 and β is a positive number of order unity. For most monoatomic gases, $\beta \approx 0.8$ fits the real situation best [21].

The relations between viscosities and collision frequency in (21) can be used to obtain an explicit relation between the latter and the variables ρ and T, which reads

$$\nu_G = \frac{1}{1-b}\frac{p}{\mu} = \nu_{G0}\rho T^{1-\beta} , \tag{25}$$

where $\nu_{G0} = \frac{RT_0^\beta}{(1-b)\mu_0}$ is a constant.

For power-law interaction potentials between particles of the type $\phi \sim r^{-(n-1)}$, one finds that

$$\beta = \frac{n+3}{2(n-1)} .$$

Here, $n = 5$ ($\beta = 1$) represents the Maxwell molecule model, $n \to \infty$ ($\beta = 1/2$) represents the hard-sphere molecule model, and $n = 23/3$ ($\beta = 4/5$) represents a molecule model that fits real gases best.

We shall assume collision frequencies of the form (25) in the remainder of this paper.

3.4. Burnett equations

Finally, let us consider the order $O(\delta)$ in (14), which yields

$$f^{(2)} = f_G^{(2)} - \frac{1}{\nu_G}\left(\frac{\partial_1 f}{\partial t} + c_i \frac{\partial f^{(1)}}{\partial x_i}\right) ,$$

where

$$\frac{\partial_1 f}{\partial t} = \frac{\partial f^{(0)}}{\partial U_r}\cdot\frac{\partial_1 U_r}{\partial t} + \frac{\partial f^{(1)}}{\partial U_r}\cdot\frac{\partial_0 U_r}{\partial t} + \frac{\partial f^{(1)}}{\partial(\nabla U_r)}\cdot\frac{\partial_0(\nabla U_r)}{\partial t} ,$$

$$\frac{\partial f^{(1)}}{\partial x_i} = \frac{\partial f^{(1)}}{\partial U_r}\cdot\frac{\partial U_r}{\partial x_i} + \frac{\partial f^{(1)}}{\partial(\nabla U_r)}\cdot\frac{\partial(\nabla U_r)}{\partial x_i} .$$

After a tedious manipulation, the expression of $f^{(2)}$ is obtained as

$$f^{(2)} = f_G^{(2)} + \frac{f_m}{\nu_G^2}\Theta , \tag{26}$$

where

$$\begin{aligned}
\Theta = &-\frac{C_iC_j}{(1-b)\rho}\frac{\partial^2\rho}{\partial x_{<i}\partial x_{j>}} + \frac{4-5b}{3(1-b)}C_i\frac{\partial^2 u_j}{\partial x_i\partial x_j} - \frac{C_i}{1-b}\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{C_iC_jC_k}{RT(1-b)}\frac{\partial^2 u_k}{\partial x_i\partial x_j}\\
&-\frac{2-b}{3(1-b)}\frac{C_iC^2}{RT}\frac{\partial^2 u_j}{\partial x_i\partial x_j} + \frac{5R}{2}\frac{\partial^2 T}{\partial x_i\partial x_i} - \frac{7-5b}{2(1-b)}\frac{C_iC_j}{T}\frac{\partial^2 T}{\partial x_i\partial x_j} - \frac{3-5b}{6(1-b)}\frac{C^2}{T}\frac{\partial^2 T}{\partial x_i\partial x_i}\\
&+\frac{C_iC_jC^2}{2RT^2}\frac{\partial^2 T}{\partial x_i\partial x_j} + \frac{C_{<i}C_{j>}}{(1-b)\rho^2}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j} + \frac{2C_j}{(1-b)\rho}\frac{\partial u_{<i}}{\partial x_{j>}}\frac{\partial\rho}{\partial x_i} - \frac{C_iC_jC_k}{(1-b)\rho RT}\frac{\partial u_{<i}}{\partial x_{j>}}\frac{\partial\rho}{\partial x_k}\\
&-\frac{5R}{2\rho}\frac{\partial T}{\partial x_i}\frac{\partial\rho}{\partial x_i} + \frac{5-7b}{2(1-b)}\frac{C_iC_j}{\rho T}\frac{\partial T}{\partial x_i}\frac{\partial\rho}{\partial x_j} + \frac{5-3b}{6(1-b)}\frac{C^2}{\rho T}\frac{\partial T}{\partial x_i}\frac{\partial\rho}{\partial x_i} - \frac{C_iC_jC^2}{2\rho RT^2}\frac{\partial T}{\partial x_i}\frac{\partial\rho}{\partial x_j}\\
&+\frac{2}{1-b}\frac{\partial u_{<i}}{\partial x_{j>}}\frac{\partial u_i}{\partial x_j} + \frac{2(3.5-\beta)}{3(1-b)}\frac{C_kC_j}{RT}\frac{\partial u_{<k}}{\partial x_{j>}}\frac{\partial u_i}{\partial x_i} - \frac{C_{<i}C_{j>}}{(1-b)RT}\frac{\partial u_k}{\partial x_j}\frac{\partial u_i}{\partial x_k} - \frac{2C_iC_j}{(1-b)RT}\frac{\partial u_{<k}}{\partial x_{j>}}\frac{\partial u_k}{\partial x_i}\\
&-\frac{2}{3(1-b)}\frac{C^2}{RT}\frac{\partial u_{<i}}{\partial x_{j>}}\frac{\partial u_i}{\partial x_j} + \frac{C_iC_jC_kC_l}{(1-b)R^2T^2}\frac{\partial u_{<k}}{\partial x_{l>}}\frac{\partial u_{<i}}{\partial x_{j>}} + \frac{1-\beta}{1-b}\frac{C_i}{T}\frac{\partial T}{\partial x_j}\frac{\partial u_i}{\partial x_j}
\end{aligned}$$

$$+\left(5+\frac{1-\beta}{1-b}\right)\frac{C_i}{T}\frac{\partial T}{\partial x_j}\frac{\partial u_j}{\partial x_i}-\frac{(15-15b)+(14-10b)(1-\beta)}{6(1-b)}\frac{C_i}{T}\frac{\partial T}{\partial x_i}\frac{\partial u_j}{\partial x_j}$$
$$-\frac{(14-7b)+2(1-\beta)}{2(1-b)}\frac{C_iC_jC_k}{RT^2}\frac{\partial T}{\partial x_i}\frac{\partial u_k}{\partial x_j}+\frac{(8.5-5b)+(2-b)(1-\beta)}{3(1-b)}\frac{C_iC^2}{RT^2}\frac{\partial T}{\partial x_i}\frac{\partial u_j}{\partial x_j}$$
$$-\frac{C_iC^2}{RT^2}\frac{\partial T}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\frac{2-b}{2(1-b)}\frac{C_iC_jC_kC^2}{R^2T^3}\frac{\partial T}{\partial x_k}\frac{\partial u_{<i}}{\partial x_{j>}}-\frac{5(1-\beta)}{2}\frac{R}{T}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i}$$
$$+\left(\frac{39}{4}+\frac{5}{2}(1-\beta)\right)\frac{C_iC_j}{T^2}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}-\left(\frac{1}{3}-\frac{5}{6}(1-\beta)\right)\frac{C^2}{T^2}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i}-\frac{8-\beta}{2}\frac{C_iC_jC^2}{RT^3}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}$$
$$+\frac{C_iC_jC^4}{4R^2T^4}\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}\,.$$

It is straightforward, but cumbersome, to show that this expression indeed satisfies the compatibility conditions (10).

The pressure tensor and heat flux at second order, the Burnett order, are computed as

$$p_{ij}^{(2)}=\sigma_{ij}^{(2)}=\frac{1}{(1-b)^2\,\nu_G^2}\varPhi_{ij}\,,\quad q_i^{(2)}=\frac{1}{(1-b)\,\nu_G^2}\varGamma_i\,, \tag{27}$$

where

$$\varPhi_{ij}=-2R^2T^2\frac{\partial^2\rho}{\partial x_{<i}\partial x_{j>}}-2b\rho R^2T\frac{\partial^2 T}{\partial x_{<i}\partial x_{j>}}+\frac{2R^2T^2}{\rho}\frac{\partial\rho}{\partial x_{<i}}\frac{\partial\rho}{\partial x_{j>}}-2R^2T\frac{\partial T}{\partial x_{<i}}\frac{\partial\rho}{\partial x_{j>}}$$
$$+2p\frac{\partial u_{<i}}{\partial x_k}\frac{\partial u_{j>}}{\partial x_k}+\frac{2-4\beta}{3}p\frac{\partial u_{<i}}{\partial x_{j>}}\frac{\partial u_k}{\partial x_k}+2\beta\,(1-b)\,\rho R^2\frac{\partial T}{\partial x_{<i}}\frac{\partial T}{\partial x_{j>}}\,,$$
$$\varGamma_i=\frac{5b-4}{3}pRT\frac{\partial^2 u_j}{\partial x_i\partial x_j}+pRT\frac{\partial^2 u_i}{\partial x_j\partial x_j}-2R^2T^2\frac{\partial u_{<j}}{\partial x_{i>}}\frac{\partial\rho}{\partial x_j}+\left(6+\beta-\frac{7}{2}b\right)pR\frac{\partial u_i}{\partial x_j}\frac{\partial T}{\partial x_j}$$
$$+\left(1+\beta+\frac{3}{2}b\right)pR\frac{\partial u_j}{\partial x_i}\frac{\partial T}{\partial x_j}+\frac{(-13-b)+(14-10b)\,(1-\beta)}{6}pR\frac{\partial u_j}{\partial x_j}\frac{\partial T}{\partial x_i}\,. \tag{28}$$

The above expressions for $\sigma_{ij}^{(2)}$ and $q_i^{(2)}$ are irreducible forms in terms of the gradients of density, velocity, and temperature.

For the special choice $b=0$, the ES-BGK model reduces to the standard BGK model, and the corresponding Burnett expressions for $\sigma_{ij}^{(2)}$ and $q_i^{(2)}$ read

$$\sigma_{ij}^{(2)}=\frac{\mu^2}{p^2}\varPhi_{ij}^{\mathrm{BGK}},\quad q_i^{(2)}=\frac{\mu^2}{p^2}\varGamma_i^{\mathrm{BGK}}\,, \tag{29}$$

where

$$\varPhi_{ij}^{\mathrm{BGK}}=-2R^2T^2\frac{\partial^2\rho}{\partial x_{<i}\partial x_{j>}}+\frac{2R^2T^2}{\rho}\frac{\partial\rho}{\partial x_{<i}}\frac{\partial\rho}{\partial x_{j>}}-2R^2T\frac{\partial T}{\partial x_{<i}}\frac{\partial\rho}{\partial x_{j>}}+2p\frac{\partial u_{<i}}{\partial x_k}\frac{\partial u_{j>}}{\partial x_k}$$
$$+\frac{2-4\beta}{3}p\frac{\partial u_{<i}}{\partial x_{j>}}\frac{\partial u_k}{\partial x_k}+2\beta\rho R^2\frac{\partial T}{\partial x_{<i}}\frac{\partial T}{\partial x_{j>}}\,,$$

$$\varGamma_i^{\mathrm{BGK}}=-\frac{4}{3}pRT\frac{\partial^2 u_j}{\partial x_i\partial x_j}+pRT\frac{\partial^2 u_i}{\partial x_j\partial x_j}-2R^2T^2\frac{\partial u_{<j}}{\partial x_{i>}}\frac{\partial\rho}{\partial x_j}+(6+\beta)\,pR\frac{\partial u_i}{\partial x_j}\frac{\partial T}{\partial x_j}$$
$$+(1+\beta)\,pR\frac{\partial u_j}{\partial x_i}\frac{\partial T}{\partial x_j}+\frac{1-14\beta}{6}pR\frac{\partial u_j}{\partial x_j}\frac{\partial T}{\partial x_i}\,. \tag{30}$$

The Burnett equations for the BGK model are available in the literature (see [13–15]). In [13, 14], only hard-sphere molecules ($\beta=1$) were considered, and our results agree with those in [13, 14], when we set $\beta=1$. The Burnett BGK equations reported in [15] are questionable: For example, since $\sigma_{ij}^{(2)}$ is a trace-free tensor, one finds that $\alpha_7/\alpha_8=-2$ should hold in (11) of [15], but the authors report $\alpha_7/\alpha_8=3.5$ instead. Similar problems arise with other coefficients. Thus, it seems that the equations in [15] are not correct.

In order to compare the Burnett equations for the ES-BGK model with the Burnett equations for the Boltzmann equation (see, e.g., [3, 22, 23]), we rewrite (27) and (28) as

$$\sigma_{ij}^{(2)}=\varpi_1\frac{\mu^2}{p}\frac{\partial u_k}{\partial x_k}\frac{\partial u_{<i}}{\partial x_{j>}}+\varpi_2\frac{\mu^2}{p}\left(-\frac{\partial}{\partial x_{<i}}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_{j>}}\right)-\frac{\partial u_k}{\partial x_{<i}}\frac{\partial u_{j>}}{\partial x_k}-2\frac{\partial u_k}{\partial x_{<j}}\frac{\partial u_{<i>}}{\partial x_{k>}}\right)$$

$$+\varpi_3 \frac{\mu^2}{\rho T} \frac{\partial^2 T}{\partial x_{<i} \partial x_{j>}} + \varpi_4 \frac{\mu^2}{\rho p T} \frac{\partial T}{\partial x_{<i}} \frac{\partial p}{\partial x_{j>}} + \varpi_5 \frac{\mu^2}{\rho T^2} \frac{\partial T}{\partial x_{<i}} \frac{\partial T}{\partial x_{j>}} + \varpi_6 \frac{\mu^2}{p} \frac{\partial u_{<k}}{\partial x_{<i>}} \frac{\partial u_{<j>}}{\partial x_{k>}} ,$$

$$q_i^{(2)} = \theta_1 \frac{\mu^2}{\rho T} \frac{\partial u_j}{\partial x_j} \frac{\partial T}{\partial x_i} - \theta_2 \frac{\mu^2}{\rho T} \left(\frac{2T}{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} + \frac{2}{3} \frac{\partial u_j}{\partial x_j} \frac{\partial T}{\partial x_i} + 2 \frac{\partial u_j}{\partial x_i} \frac{\partial T}{\partial x_j} \right) \tag{31}$$

$$-\theta_3 \frac{\mu^2}{\rho p} \frac{\partial u_{<i}}{\partial x_{j>}} \frac{\partial p}{\partial x_j} + \theta_4 \frac{\mu^2}{\rho} \frac{\partial^2 u_{<i}}{\partial x_j \partial x_{j>}} + \theta_5 \frac{3\mu^2}{\rho T} \frac{\partial u_{<i}}{\partial x_{j>}} \frac{\partial T}{\partial x_j} ,$$

where the so-called Burnett coefficients ϖ_α and θ_α are given by

$$\varpi_1 = \frac{4}{3} \left(\frac{7}{2} - \beta \right) , \ \varpi_2 = 2, \ \varpi_3 = 2(1-b), \ \varpi_4 = 0, \ \varpi_5 = 2\beta(1-b), \ \varpi_6 = 8 , \tag{32}$$

$$\theta_1 = \frac{5}{3} (1-b)^2 \left(\frac{7}{2} - \beta \right) , \ \theta_2 = \frac{5}{2} (1-b)^2 , \ \theta_3 = 2(1-b), \ \theta_4 = 2(1-b), \ \theta_5 = \frac{2}{3} (1-b) \left(7 + \beta - \frac{7}{2} b \right) .$$

In particular we are interested in the case in which the Prandtl number has its proper value $\mathrm{Pr} = 2/3$, which is the case for $b = -1/2$. Then, the coefficients ϖ_α and θ_α have the values

$$\varpi_1 = \frac{4}{3} \left(\frac{7}{2} - \beta \right) , \ \varpi_2 = 2, \ \varpi_3 = 3, \ \varpi_4 = 0, \ \varpi_5 = 3\beta, \ \varpi_6 = 8 ,$$

$$\theta_1 = \frac{15}{4} \left(\frac{7}{2} - \beta \right) , \ \theta_2 = \frac{45}{8}, \ \theta_3 = 3, \ \theta_4 = 3, \ \theta_5 = \frac{35}{4} + \beta . \tag{33}$$

For the special case of Maxwell molecules, for which $\beta = 1$, the above expressions (31) and (33) are identical to the expressions given in [22]. For any molecule model with power-law interaction potentials, the above expressions are identical to the expressions given in [3][1], which are the Burnett equations of the Boltzmann equation for a first-order solution of the resulting integral equation in terms of Sonine polynomials.

Table 1 shows the values of the Burnett coefficients for different values of n (or β) compared with values obtained by Reinecke and Kremer [23] from high-accuracy computations for the full Boltzmann equation, i.e., a fifth-order expansion in Sonine polynomials. Note that the expansion in Sonine polynomials is necessary to solve the integral equations that appear in the Chapman–Enskog expansion of the full Boltzmann equation, so that in this case the Chapman–Enskog expansion of a given order can only be computed in an approximate manner. It must be noted here that the fifth-order expansion yields very accurate results, so that the results in [23] can be considered as almost exact.

For the ES-BGK model this additional difficulty does not arise due to the simpler collision term, and the Chapman–Enskog expansion of a given order can be computed exactly, without further approximations. However, we emphasize that the ES-BGK collision term itself is an approximation of the Boltzmann collision term and does not preserve the full physics of the Boltzmann equation.

Indeed, from the values in Table 1 it becomes apparent that the Burnett equations from the ES-BGK model agree with the Burnett equations from the Boltzmann equation only for Maxwell molecules ($\beta = 1$). For other molecule models, the coefficients are close to those of the full Boltzmann equation, but different nevertheless. Thus we can state that the ES-BGK model agrees with the Boltzmann equation for Maxwell molecules up to second order in the Knudsen number, while for more realistic molecule models (e.g. $\beta = 4/5$), some differences occur.

From numerical simulations of various BGK models [11], we conclude that the differences between the ES-BGK model and the Boltzmann equation are at least partly related to the collision frequency, which is assumed to be independent of the microscopic velocity in the ES-BGK model. However, it is easy to see that the collision frequency is independent of the microscopic velocity only for Maxwell molecules, but not for other types of interaction [11]. Thus, it might well be that an ES model with a velocity-dependent collision frequency will yield Burnett coefficients closer to those of the Boltzmann equation. Currently, we are developing such a model.

[1] Take into account that, by (24), $\frac{T}{\mu} \frac{d\mu}{dT} = \beta$.

Table 1. Burnett coefficients ϖ_1 to ϖ_6 and θ_1 to θ_5 for different power indices n and $b = -1/2$. The values in brackets are those from the full Boltzmann equation [23], for which values for $n = 23/3$ are interpolated

n	β	ϖ_1	ϖ_2	ϖ_3	ϖ_4	ϖ_5	ϖ_6
5	1	10/3	2	3	0	3	8
		(10/3)	(2)	(3)	(0)	(3)	(8)
23/3	4/5	3.60	2	3	0	2.40	8
		(3.600370)	(2.004303)	(2.760746)	(0.253684)	(1.783502)	(7.748040)
∞	1/2	4.00	2	3	0	1.50	8
		(4.057097)	(2.028549)	(2.415493)	(0.680112)	(0.232355)	(7.419524

n	β	θ_1	θ_2	θ_3	θ_4	θ_5
5	1	75/8	45/8	3	3	39/4
		(75/8)	(45/8)	(3)	(3)	(39/4)
23/3	4/5	10.13	5.625	3	3	9.55
		(10.159655)	(5.655703)	(3.014430)	(2.760746)	(9.018759)
∞	1/2	11.25	5.625 6 3	3	3	9.25
		(11.652480)	(5.826240)	(3.095605)	(2.415493)	(8.100700)

4. Linear stability analysis

It is well known that the Burnett equations that follow from the Boltzmann equation are linearly unstable [12]. In this section, we consider the linear stability of the Burnett equations for the ES-BGK model in which we restrict ourselves to one-dimensional processes. We follow the procedure for stability analysis outlined in [12].

We are interested only in processes with small deviations from an equilibrium state ρ_0, T_0, and $u_{i,0} = 0$. Then, we can introduce dimensionless variables $\hat{\rho}$, $\hat{T}$, $\hat{u}_i$, $\hat{\sigma}_{ij}$, and $\hat{q}_i$ by

$$\rho = \rho_0 \left(1 + \hat{\rho}\right), \quad T = T_0 \left(1 + \hat{T}\right), \quad u_i = \sqrt{RT_0}\hat{u}_i, \quad \sigma_{ij} = p_0 \hat{\sigma}_{ij}, \quad q_i = \rho_0 \left(RT_0\right)^{3/2} \hat{q}_i \, .$$

The new variables measure the deviation from a global equilibrium state, and, in the linear case, are much smaller than unity.

Moreover, we use the mean free path l to introduce non-dimensional space and time variables $\hat{x}_i$ and $\hat{t}$ as

$$x_i = l\hat{x}_i, \quad t = \frac{l}{\sqrt{RT_0}}\hat{t} = \frac{\mu}{p_0}\hat{t}, \quad \text{where} \quad l = \frac{\mu\sqrt{RT_0}}{p_0} \, .$$

Linearization in the new variables yields the dimensionless linearized three -dimensional Burnett equations as

$$\begin{aligned} &\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial \hat{u}_i}{\partial \hat{x}_i} = 0 \, , \\ &\frac{\partial \hat{u}_i}{\partial \hat{t}} + \frac{\partial \hat{\rho}}{\partial \hat{x}_i} + \frac{\partial \hat{T}}{\partial \hat{x}_i} - \frac{\partial^2 \hat{u}_i}{\partial \hat{x}_j \partial \hat{x}_j} - \frac{1}{3}\frac{\partial^2 \hat{u}_j}{\partial \hat{x}_j \partial \hat{x}_i} - \frac{4}{3}\frac{\partial^3 \hat{\rho}}{\partial \hat{x}_j \partial \hat{x}_j \partial \hat{x}_i} - \frac{4b}{3}\frac{\partial^3 \hat{T}}{\partial \hat{x}_j \partial \hat{x}_j \partial \hat{x}_i} = 0 \, , \\ &\frac{3}{2}\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial \hat{u}_i}{\partial \hat{x}_i} - \frac{5\left(1-b\right)}{2}\frac{\partial^2 \hat{T}}{\partial \hat{x}_i \partial \hat{x}_i} - \frac{(1-b)(1-5b)}{3}\frac{\partial^3 \hat{u}_j}{\partial \hat{x}_i \partial \hat{x}_i \partial \hat{x}_j} = 0 \, . \end{aligned} \tag{34}$$

Note that the value of the viscosity exponent β plays no role in the linear equations. The hats will be omitted in what follows in order to simplify the notation.

In the case of one-dimensional processes for which all variables depend only on $x_1 = x$, and for which $u_i = \{u, 0, 0\}$, the above equations reduce further to

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} = 0 \, ,$$

$$\frac{\partial u}{\partial t}+\frac{\partial \rho}{\partial x}+\frac{\partial T}{\partial x}-\frac{4}{3}\frac{\partial^2 u}{\partial x^2}-\frac{4}{3}\frac{\partial^3 \rho}{\partial x^3}-\frac{4b}{3}\frac{\partial^3 T}{\partial x^3}=0\,,$$
$$\frac{3}{2}\frac{\partial T}{\partial t}+\frac{\partial u}{\partial x}-\frac{5(1-b)}{2}\frac{\partial^2 T}{\partial x^2}-\frac{(1-b)(1-5b)}{3}\frac{\partial^3 u}{\partial x^3}=0\,. \tag{35}$$

We now assume plane wave solutions of the form

$$\rho=\tilde{\rho}\exp(\lambda t+ikx)\,,\quad T=\tilde{T}\exp(\lambda t+ikx)\,,\quad u=\tilde{u}\exp(\lambda t+ikx)\,,$$

where k is the positive real wave number, while λ is a complex number, whose real part can be interpreted as the negative of the damping of the wave, and whose imaginary part is the frequency of the wave. Stability of the system requires negative damping, so that $\mathrm{Re}\,\lambda \le 0$ must hold for all possible values of k.

Substitution of these solutions into (35) yields a homogeneous algebraic system for the amplitudes $\tilde{\rho}$, $\tilde{u}$, and $\tilde{T}$ that reads

$$\lambda\cdot\tilde{\rho}+ik\cdot\tilde{u}=0\,,$$
$$\left(ik+\frac{4}{3}ik^3\right)\cdot\tilde{\rho}+\left(\lambda+\frac{4}{3}k^2\right)\cdot\tilde{u}+\left(ik+\frac{4b}{3}ik^3\right)\cdot\tilde{T}=0\,, \tag{36}$$
$$\left(ik+\frac{(1-b)(1-5b)}{3}ik^3\right)\cdot\tilde{u}+\left(\frac{3}{2}\lambda+\frac{5(1-b)}{2}k^2\right)\cdot\tilde{T}=0\,.$$

The condition for the existence of a nontrivial solution of above system is that the determinant of the system equals zero, that is,

$$\lambda^3+\frac{9-5b}{3}k^2\lambda^2+k^2\lambda\cdot\left[\frac{5}{3}+k^2\frac{10b^2-24b+34}{9}+k^4\frac{8b(1-b)(1-5b)}{27}\right]$$
$$+\frac{5(1-b)}{3}k^4\left(1+\frac{4}{3}k^2\right)=\mathit{\Omega}(\lambda,\,k)=0\,. \tag{37}$$

Note that the ES-BGK model requires that $-1/2 \le b \le 1$. We will divide the whole range of b into three parts to discuss the roots of the above cubic equation for λ, and the linear stability of one-dimensional processes.

• **Part I** $(-1/2 \le b < 0$ **and** $1/5 < b < 1)$

We follow the line of arguments of [12]. For values of b in this range, it is easy to see that $\mathit{\Omega}(\lambda=0,\,k)>0$ and $\mathit{\Omega}(\lambda\to\infty,\,k)\to+\infty$, while for very large values of k,

$$\mathit{\Omega}\left(\lambda=k^2,\,k\right)\approx\frac{8b(1-b)(1-5b)}{27}k^8<0$$

holds. Therefore, (37) has at least two positive real roots λ. It follows that the Burnett equations for the ES-BGK model are linearly unstable for values of b in the intervals $-1/2 \le b < 0$ and $1/5 < b < 1$.

• **Part II** $(b=1)$

In this particular case, (37) simplifies to

$$\lambda^3+\frac{4}{3}k^2\lambda^2+k^2\lambda\left[\frac{5}{3}+k^2\frac{20}{9}\right]=0\,.$$

The three roots for this cubic equation are

$$\lambda_1=0,\quad \lambda_{2,\,3}=-\frac{2}{3}k^2\pm i\sqrt{\frac{16}{9}k^4+\frac{5}{3}k^2}\,.$$

Therefore, $\mathrm{Re}\,\lambda_{1,\,2,\,3}\le 0$, and the Burnett equations for the ES-BGK model are linearly stable for $b=1$. Note that by (21) this case corresponds to a gas with infinite viscosity and $\mathrm{Pr}\to\infty$

• **Part III** $(0 \le b \le 1/5)$

A cubic equation has either three real roots, or one real root and two conjugate complex roots [24]. We shall analyze the roots of (37) for $0 \le b \le 1/5$.

a. Three real roots

Assume that the three roots of (37) are all real. Since the function $\Omega\,(\lambda,\,k)$ (the left hand side of (37)) increases when λ increases ($\lambda \geq 0$ assumed), and $\Omega\,(\lambda = 0,\,k) > 0$, it follows that (37) cannot have positive roots, and that all three real roots (if they exist) will be negative.

b. One real root and two conjugate complex roots

In this case, we assume that the three roots of (37) are one real value λ_1 and two conjugate complex values $\lambda_{2,\,3} = \lambda_R \pm i\lambda_i$, where λ_R and λ_i themselves are real numbers. Since, for $\lambda \geq 0$, the function $\Omega\,(\lambda,\,k)$ increases when λ increases, and since $\Omega\,(\lambda = 0,\,k) > 0$, λ_1 must be negative. Furthermore, from (37), we obtain by inserting $\lambda = \lambda_R + i\lambda_i$

$$\lambda_R^3 - 3\lambda_R\lambda_i^2 + \frac{9-5b}{3}k^2\left(\lambda_R^2 - \lambda_i^2\right) + k^2\lambda_R\left[\frac{5}{3} + k^2\frac{10b^2 - 24b + 34}{9} + k^4\frac{8b(1-b)(1-5b)}{27}\right] + $$
$$+\frac{5(1-b)}{3}k^4\left(1 + \frac{4}{3}k^2\right) = 0\,,$$
$$3\lambda_R^2 - \lambda_i^2 + \frac{9-5b}{3}k^2 2\lambda_R + k^2\left[\frac{5}{3} + k^2\frac{10b^2 - 24b + 34}{9} + k^4\frac{8b(1-b)(1-5b)}{27}\right] = 0\,. \tag{38}$$

Eliminating λ_i between these two equations yields

$$\lambda_R^3 + \zeta_1\lambda_R^2 + \zeta_2\lambda_R + \zeta_3 = 0\,, \tag{39}$$

where the k-dependent coefficients are given by

$$\zeta_1 = \frac{9-5b}{3}k^2\,,$$
$$\zeta_2 = \left(\frac{9-5b}{6}\right)^2 k^4 + \frac{k^2}{4}\left[\frac{5}{3} + k^2\frac{10b^2 - 24b + 34}{9} + k^4\frac{8b(1-b)(1-5b)}{27}\right]\,,$$
$$\zeta_3 = \frac{b(9-5b)(1-b)(1-5b)}{81}k^8 + \frac{123 - 163b + 105b^2 - 25b^3}{108}k^6 + \frac{15-5b}{36}k^4\,.$$

Since the coefficients ζ_1, ζ_2, and ζ_3 are positive for $0 \leq b \leq 1/5$, there are no positive roots of (39) and it follows that $\mathrm{Re}\,\lambda \leq 0$.

Thus, $\mathrm{Re}\,\lambda \leq 0$ holds for all values of b in the interval and we conclude that the Burnett equations for the ES-BGK model are linearly stable for $0 \leq b \leq 1/5$, that is, for Prandtl numbers $1 \leq \mathrm{Pr} \leq 5/4$. For all other values of b (or Pr) except $\mathrm{Pr} \to \infty$, however, the Burnett equations for the ES-BGK model are linearly unstable – this includes the physically relevant case of $\mathrm{Pr} = 2/3$.

The stability of the BGK-Burnett equations, for which $\mathrm{Pr} = 1$, can be found in the literature [16]. Our analysis shows that stability can be obtained for a wider range of values for the Prandtl number.

5. Conclusion

The Burnett equations for the ES-BGK model with power-law interaction potentials have been computed by the Chapman–Enskog method to second order. When the Prandtl number is adjusted to its proper value $\mathrm{Pr} = 2/3$ ($b = -1/2$), the ES-BGK Burnett equations are found to be identical to the Burnett equations for the Boltzmann equation only in the case of Maxwell molecules, while the Burnett coefficients exhibit some differences for other interaction types (e.g. $n = 23/3$ or $n \to \infty$ – hard spheres). This indicates that computations of processes with the ES-BGK equation will show some discrepancies to computations that are based on the Boltzmann equation. For processes with Knudsen numbers that are not too large, these differences might be small. Since our analysis is based on an expansion in terms of the Knudsen number only to second order, we cannot make any prediction as to how close the ES-BGK model will be to the Boltzmann equation when the Knudsen number is large.

Our analysis of the linear stability of the ES-BGK Burnett equations has revealed that the stability depends on the value of the Prandtl number. For $1 \leq \mathrm{Pr} \leq 5/4$ and for $\mathrm{Pr} \to \infty$, the ES-BGK Burnett equations are linearly stable, while they are linearly unstable for $2/3 < \mathrm{Pr} < 1$ and $5/4 < \mathrm{Pr} < \infty$. It follows that higher order Chapman–Enskog expansions do not necessarily lead to unstable equations.

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