SOME REMARKS ON THE EQUATIONS OF BURNETT AND GRAD

HENNING STRUCHTRUP*

Abstract. Both, Grad and Burnett, derived sets of equations from the Boltzmann equation, which improve the classical laws of Navier-Stokes and Fourier for the description of rarefied gases, i.e. gases with Knudsen numbers above 0.01. Using results of other authors, it is shown that both sets of equations are closer related then is commonly thought - indeed, the Burnett equations can be derived from Grad's equations by the so-called Maxwellian iteration. This derivation allows to identify the proper form of the Burnett equations in non-inertial frames. Moreover, Grad's equations with more than 13 moments can describe linear boundary layers while these are not among the phenomena which can be described by Burnett's equations.

1. Introduction. Processes in rarefied gases are well described by the Boltzmann equation [5, 6]. The numerical solution of the Boltzmann equation, either directly [17] or via the Direct Simulation Monte Carlo (DSMC) method [3], is very time expensive [13], and there is a strong desire for accurate models which allow the calculation of processes in rarefied gases at lower computational cost.

There are two well-known approaches towards this goal, the Chapman-Enskog method [5, 6, 9] and the method of moments of Grad [8, 16]. In the present paper we shall try to show the strong relations between these two approaches. Moreover, we will briefly discuss the limits and advantages of the methods. It will become clear that the Burnett equations, obtained by the second order Chapman-Enskog expansion, can be obtained from the Grad equations as well. Mainly our discussion relies on the use of the Maxwellian iteration of Truesdell and Ikenberry [29, 30], in the interpretation of Müller [15] and Reinecke and Kremer [18, 19].

In the Chapman-Enskog method, the phase density is expanded in powers of the Knudsen number Kn, defined as the ratio between the mean free path of the molecules and the relevant macroscopic length scale. To zeroth order the expansion yields the Euler equations, the first order correction results in the equations of Navier-Stokes and Fourier, and the second order expansion yields the Burnett equations. The equations of Navier-Stokes and Fourier cease to be accurate for Knudsen numbers above 0.01, and it is believed that the Burnett equations are valid for larger Knudsen numbers.

However, the Burnett equations might not be the appropriate tool for the description of processes in rarefied gases [9]. It is well known that the Burnett equations are linearly ill-posed so that waves of small wave length are not damped but grow at unbounded rate [4, 21]. This corresponds to violations of the second law, which are also observed in the case

---

*Department of Mechanical Engineering, University of Victoria, P.O. Box 3055 STN CSC, Victoria BC V8W 3P6, Canada (struchtr@me.uvic.ca).

of strong gradients in stationary problems [7, 25]. Despite these problems, the Burnett equations are still discussed in the literature [31, 32]. In the last years attempts were made to obtain stable equations by adding terms of the next order in the expansion (Super-Burnett) [34, 22, 10]. With these augmented Burnett equations, it is possible to compute shock structures for any Mach number [2], while the traditional Burnett equations cannot resolve the whole extend of the shock [31]. The Burnett equations cannot describe linear boundary layers, but their non-linearities allow for a boundary layer analysis [13]. For a proper description of the boundaries, however, the Burnett equations should be supplemented by an appropriate boundary layer theory [5, 11]. Moreover, questions arise about the form the Burnett equations should have in non-inertial frames.

In the method of moments of Grad [8], the Boltzmann equation is replaced by a set of moment equations, - first order partial differential equations for the moments of the phase density. Which and how many moments are needed depends on the particular process, but experience shows that the number of moments must be increased with increasing Knudsen number [16, 23]. For the closure of the equations, the phase density is approximated by an expansion in Hermite polynomials about the equilibrium distribution (the local Maxwellian), where the coefficients are related to the moments.

Only few moments have an intuitive physical meaning, i.e. density ϱ, momentum density ϱv_i, energy density ϱe, heat flux q_i and pressure tensor p_{ij}. This set of 13 moments[1] forms the basis of Grad's famous 13 moment equations [8] which are commonly discussed in textbooks [12]. However, the 13 moment set does not allow the computation of boundary layers [24] and leads to shock structures with discontinuities for Mach numbers above 1.65 [16]. With increasing number of moments, one can compute accurate boundary layers [20, 28] and smooth shock structures up to higher Mach numbers [33, 1]. As becomes evident from the cited literature, one has to face hundreds of moment equations in case of large Mach or Knudsen numbers.

Basically, Grad's equations are of symmetric hyperbolic type for a certain range of values for the moments. The possible loss of hyperbolicity is discussed in [16] for the 13 field case, but was never considered for systems with more variables. Thus, it is not sure, whether the increase of the moment number helps to maintain hyperbolicity. In the papers cited above, loss of hyperbolicity was not noticed.

The sets with increasing number of moments allow for an easy check of their quality: if in the simulation of a given process the result remains almost unchanged for an increased number of moments, than the previous set of equations was suitable for that particular process. If, however, the

---

[1] The pressure tensor is symmetric, and its trace is related to the thermal energy by the ideal gas law.

result changes considerably, one has to increase the number of moments further, until the results cease to change.

A similar test is not possible for the Burnett equations, since the higher order Chapman-Enskog expansions are quite involved and were never performed above the Super-Burnett level. Thus, one cannot test the Burnett equations against the higher order expansions, and has to compare their predictions either with solutions of the moment equations or direct solutions of the Boltzmann equation. Lacking a test of their accuracy within the framework of the Chapman-Enskog expansion itself, it might happen that the Burnett equations will be applied outside their proper range.

While the derivation of both sets of equations seems to be quite independent, they are closer related than one may think. Indeed, the Burnett equations can be derived via a Maxwellian iteration from certain sets of moment equations via the Maxwellian iteration of Ikenberry and Truesdell [29, 30]. In this method, one needs not to perform Grad's closure procedure for the moment equations, but starts with an infinite set of unclosed equations - called transfer equations. Expressions for heat flux and pressure tensor follow from an iteration procedure on this infinite set.

The zeroth order iteration simply gives the Euler equations, the first order iteration involves 13 transfer equations[2] and yields the laws of Navier Stokes and Fourier. The next step relies on 26 equations, and yields basically the Burnett equations plus some third order terms, which should appear in the Super-Burnett case of the Chapman-Enskog expansion. If the corresponding sets of Grad's moment equations (e.g. with 13 or 26 moments) are considered instead of the transfer equations, the iteration should lead to the same constitutive equations.

Thus, in the Maxwell iteration, higher orders of the constitutive equations need the input of more and more transfer (or moment) equations and one might think that higher order Chapman-Enskog expansions are somehow equivalent to Grad's equations with more and more moments. This, however, is not so: the Burnett equations can be obtained already from a second iteration of Grad's set of 13 moment equations, and higher moment or transfer equations are not needed for their derivation. This fact was recognized by Reinecke and Kremer [19] who could also derive the Burnett equations for general classes of molecular interactions from extended sets of moments. Neither Reinecke and Kremer nor we in this short comment can answer the question why the two sets - Burnett equations and Grad's 13 moment equations - are so closely related.

The close relation between Burnett and Grad's 13 moment equations suggests that both sets have the same range of applicability. This means in particular that those processes which are known to be best described by a large number of moments probably cannot be described accurately by the Burnett equations.

---

[2] The numbers of equations given here refer to the special case of Maxwellian molecules.

The second iteration of the 13 moment system can already be found in Müller's book [15] where it was used as a shortcut for an argument on the frame dependence of heat flux and stress tensor that he gave before on basis of the original Maxwell iteration [14]. As it seems, he did not recognize that his equations are basically equal to the Burnett equations. Since he gave his equations in non-inertial frames, they can be used in order to formulate the Burnett equations in non-inertial frames. This will be done in the next section of this paper. The third section will briefly show that the Burnett equations as well as Grad's 13 moment case do not exhibit linear boundary layers in the simulation of stationary heat transfer, while Grad's 26 moment case does. Numerical results for even higher moment numbers are given.

2. 13 moment equations and Burnett equations. We do not consider any details of the derivation of the 13 moment equations, see [8, 15] for reference. The moments under consideration are the density ρ, the momentum density ρv_i, the energy density ρe, heat flux q_i and traceless part of the pressure tensor $p_{\langle ij\rangle}$. The energy density is related to the temperature by $\rho e = \frac{3}{2}\rho\frac{k}{m}T + \frac{\rho}{2}v^2$ where k is Boltzmann's constant and m denotes the mass of a gas particle. The trace of the pressure tensor gives the pressure by $p_{kk} = 3p = 3\rho\frac{k}{m}T$. The moment equations for the 13 moments read in a non-inertial frame

$$
\begin{aligned}
&\frac{d\rho}{dt} + \rho\frac{\partial v_k}{\partial x_k} = 0\\
&\rho\frac{dv_i}{dt} + \frac{\partial p_{ik}}{\partial x_k} = \rho\left(f_i + i_i\right)\\
&\frac{3}{2}\rho\frac{k}{m}\left[\frac{dT}{dt} + \frac{2}{3}T\frac{\partial v_k}{\partial x_k}\right] + \frac{\partial q_k}{\partial x_k} + p_{\langle ij\rangle}\frac{\partial v_i}{\partial x_j} = 0\\
&\frac{dp_{\langle ij\rangle}}{dt} + p_{\langle ij\rangle}\frac{\partial v_k}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - 4p_{\langle k\langle i\rangle}W_{j\rangle k} + 2p_{\langle k\langle i\rangle}\frac{\partial v_{j\rangle}}{\partial x_k}\\
&\qquad\qquad + 2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -\frac{p}{\mu}p_{\langle ij\rangle}\\
&\frac{dq_i}{dt} + q_k\frac{\partial v_i}{\partial x_k} - 2q_kW_{ik} + \frac{5}{2}\frac{k}{m}p\frac{\partial T}{\partial x_i} + \frac{7}{5}q_i\frac{\partial v_k}{\partial x_k} + \frac{k}{m}T\frac{\partial p_{\langle ik\rangle}}{\partial x_k}\\
&\quad + \frac{7}{2}\frac{k}{m}p_{\langle ik\rangle}\frac{\partial T}{\partial x_k} - \frac{p_{\langle ik\rangle}}{\rho}\frac{\partial p_{\langle kn\rangle}}{\partial x_n} + \frac{4}{5}q_k\frac{\partial v_{(i}}{\partial x_{k)}} - \frac{p_{\langle ik\rangle}}{\rho}\frac{\partial p}{\partial x_k} = -\frac{2}{3}\frac{p}{\mu}q_i.
\end{aligned}
\tag{1}
$$

Here, f_i are the body forces (e.g. gravity), i_i stands for the inertial forces acting on the gas in a non-inertial frame, and W_{ik} is the (antisymmetric) matrix of the angular velocities of the non-inertial frame. Indices in round brackets indicate the symmetric part of a tensor and indices in angular brackets denote the trace-free symmetric part of a tensor; moreover, $\frac{d}{dt} = \frac{\partial}{\partial t} + v_k\frac{\partial}{\partial x_k}$.

μ is the viscosity which is linear in the temperature for Maxwell molecules, $\frac{d\mu}{dT} = \frac{\mu}{T}$. The first step of the iteration will show that μ is the viscosity indeed. The first iterates $p^{(1)}_{\langle ij\rangle}$, $q^{(1)}_i$ are obtained by inserting the equilibrium values - computed from a local Maxwellian - of heat flux and pressure tensor into the left hand sides of $(1)_{4,5}$ and solving for the first iterates which stand on the right. The equilibrium values are $p^{(0)}_{\langle ij\rangle} = 0$ and $q^{(0)}_i = 0$ and we obtain the first iterates as

$$p^{(1)}_{\langle ij\rangle} = -2\mu S_{ij}, \qquad q^{(1)}_i = -\frac{15}{4}\frac{k}{m}\mu\frac{\partial T}{\partial x_i} \quad \text{where} \quad S_{ij} = \frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\ . \tag{2}$$

These are just the laws of Navier-Stokes and Fourier with the viscosity μ and the heat conductivity $\lambda = \frac{15}{4}\frac{k}{m}\mu$. Note that these constitutive laws are linear in the viscosity μ.

Now we turn our attention to the second step of the iteration. We insert the first iterates (2) into the left hand sides of $(1)_{4,5}$ and solve for the second iterates on the right hand side to obtain

$$\begin{aligned} p^{(2)}_{\langle ij\rangle} = {} & -2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + 2\frac{\mu}{p}\left[\frac{d\mu S_{ij}}{dt} + 2\mu S_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} - 4\mu S_{k\langle i}W_{j\rangle k}\right] \\ & + 2\frac{\mu^2}{p}S_{ij}\frac{\partial v_k}{\partial x_k} + 3\frac{k}{m}\frac{\mu}{p}\frac{\partial}{\partial x_{\langle i}}\left(\mu\frac{\partial T}{\partial x_{j\rangle}}\right) \end{aligned} \tag{3}$$

$$\begin{aligned} q^{(2)}_i = {} & -\alpha\mu\frac{\partial T}{\partial x_i} + \frac{3}{2}\frac{\alpha\mu}{p}\left[\frac{d}{dt}\left(\mu\frac{\partial T}{\partial x_i}\right) + \mu\frac{\partial T}{\partial x_k}\frac{\partial v_i}{\partial x_k} - 2\mu\frac{\partial T}{\partial x_k}W_{ik}\right] \\ & + \frac{5}{2}\frac{\alpha\mu^2}{p}\frac{\partial v_k}{\partial x_k}\frac{\partial T}{\partial x_i} + 4\frac{\alpha\mu^2}{p}S_{ik}\frac{\partial T}{\partial x_k} + 3\frac{\mu}{\rho}\frac{\partial\mu S_{ik}}{\partial x_k} \\ & - 3\frac{\mu^2}{p\rho}S_{ik}\frac{\partial p}{\partial x_k} + 6\frac{\mu^2}{p\rho}S_{ik}\frac{\partial\mu S_{kn}}{\partial x_n} \end{aligned} \tag{4}$$

where $\alpha = \frac{15}{4}\frac{k}{m}$. The second iteration adds quadratic and cubic terms in the viscosity, which plays here the role of a small parameter[3]. Higher order iterations will be obtained in the same manner, i.e. insertion of the last iterate on the left hand side, and solving for the new iterate on the right. Obviously, the higher iterates will contain terms with higher powers in μ. The third iterates, for instance, will recover the same second order terms as in the second iterates (3, 4), but will have additional third and higher order terms. Thus, the second iterates given above are only accurate within terms of second order, and we may drop the third order term in (4).

In order to show the strong similarity between Maxwell iteration and the Chapman-Enskog method, we have a look on the iterative procedure

---

[3] Indeed, in a dimensionless formulation, μ would be replaced by the Knudsen number *Kn*.

from another viewpoint. For this, we write pressure tensor and heat flux as a series in powers of the viscosity, just as one writes a similar series for the phase density in the Chapman-Enskog method [6],

$$p_{\langle ij\rangle}=P^{(0)}_{\langle ij\rangle}+\mu P^{(1)}_{\langle ij\rangle}+\mu^2 P^{(2)}_{\langle ij\rangle}+\ldots,\qquad q_i=Q^{(0)}_i+\mu Q^{(1)}_i+\mu^2 Q^{(2)}_i+\ldots \tag{5}$$

This ansatz is plugged in the moment equations for the two quantities, and $P^{(n)}_{\langle ij\rangle}$ and $Q^{(n)}_i$ are determined by equating all members with the same powers of μ. If one considers the series up to the n-th power of μ, the result will agree with the n-th Maxwell iteration in all members up to the order μ^n. In the result from the Maxwell iteration one will have additional terms with higher powers, which are subject to change in the next step, and will be ignored.

For the following discussion it will be important that the right hand sides of (3, 4) form frame dependent objective tensors. See the book [15] by Müller for details, where transformations in non-inertial frames and transformation properties of tensors are carefully discussed.

Using the linear dependence of μ on the temperature one can rewrite the second iterates without the third order term as

$$\begin{aligned}p^{(2)}_{\langle ij\rangle}=&-2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+\frac{10}{3}\frac{\mu^2}{p}S_{ij}\frac{\partial v_k}{\partial x_k}+2\frac{\mu^2}{p}\left[\frac{dS_{ij}}{dt}-2S_{k\langle i}\frac{\partial v_k}{\partial x_{j\rangle}}\right]\\&+3\frac{\mu^2}{\rho T}\frac{\partial^2 T}{\partial x_{\langle i}x_{j\rangle}}+3\frac{\mu^2}{\rho T^2}\frac{\partial T}{\partial x_{\langle i}}\frac{\partial T}{\partial x_{j\rangle}}\\&+8\frac{\mu^2}{p}\left[S_{k\langle i}S_{j\rangle k}-S_{k\langle i}W_{j\rangle k}\right]+\underline{2\frac{\mu^2}{pT}S_{ij}\left[\frac{dT}{dt}+\frac{2}{3}T\frac{\partial v_k}{\partial x_k}\right]}\end{aligned}\tag{6}$$

$$\begin{aligned}q^{(2)}_i=&-\alpha\mu\frac{\partial T}{\partial x_i}+\frac{5}{2}\frac{\alpha\mu^2}{p}\frac{\partial v_k}{\partial x_k}\frac{\partial T}{\partial x_i}+\frac{3}{2}\frac{\alpha\mu^2}{p}\left[\frac{d}{dt}\frac{\partial T}{\partial x_i}\right.\\&\left.-\frac{\partial v_k}{\partial x_i}\frac{\partial T}{\partial x_k}-2W_{ik}\frac{\partial T}{\partial x_k}\right]-3\frac{\mu^2}{p\rho}S_{ik}\frac{\partial p}{\partial x_k}+3\frac{\mu^2}{\rho}\frac{\partial S_{ik}}{\partial x_k}\\&+\frac{39}{5}\frac{\alpha\mu^2}{p}S_{ik}\frac{\partial T}{\partial x_k}+\underline{\frac{3}{2}\frac{\alpha\mu^2}{pT}\frac{\partial T}{\partial x_i}\left[\frac{dT}{dt}+\frac{2}{3}T\frac{\partial v_k}{\partial x_k}\right]}\end{aligned}\tag{7}$$

By means of the energy balance $(1)_3$ we see that the underlined terms in (6, 7) are of third order in μ as well so that they can also be neglected. These terms are an objective tensor and an objective vector, respectively, so that the remaining terms on the right hand side of (6, 7) still form objective tensors.

Here, apart from the terms which contain W_{ik}, we have recovered the Burnett equations for Maxwellian molecules [6, 9]. The additional frame dependent terms ensure the proper transformation of the equations in non-inertial frames. Since these terms introduce an explicit frame dependence, the principle of material frame indifference is not valid here: see Müllers discussion of these terms in [15, 14], where he shows that the frame dependence of the heat flux is due to the Coriolis force which acts on the particles during their free flight between collisions.

Summarizing we state that one can derive the Burnett equations including their corrections for non-inertial frames from the 13 moment equations of Grad.

3. Stationary heat transfer. In order to show the influence of higher moments on the quality of simulations, we study briefly the one-dimensional stationary heat transfer problem with 13 and 26 moments. We consider a gas at rest between two rigid walls of distance L with the wall temperatures T_0, T_L.

We start with the stationary state of the 13 moment case, where the moment equations for energy, pressure tensor and heat flux read in dimensionless form, see [24] or [28] for details,

$$\begin{aligned} \frac{\partial q}{\partial x} &= 0 \\ \frac{8}{15}\frac{\partial q}{\partial x} &= -\frac{1}{Kn}\frac{(1-\sigma)\,\sigma}{T} \\ \frac{\partial}{\partial x}\left(5T + 2T\sigma\right) &= -\frac{4}{3Kn}\frac{(1-\sigma)q}{T}\ . \end{aligned}$$

Here, $q = q_1$, $\sigma = p_{\langle 11\rangle}$ and $Kn = \mu\left(T_0\right)\frac{\sqrt{\frac{k}{m}T_0}}{Lp_0}$ is the Knudsen number. The first two equations show that the anisotropic stress vanishes, $\sigma = 0$, and the equations reduce simply to the law of Fourier

$$q = -\frac{15}{4}KnT\frac{\partial T}{\partial x} = const. \tag{8}$$

In the case under consideration, the Burnett equations reduce to the Fourier law (8) with $\sigma = 0$, so that both, Burnett's and Grad's equations lead to the same problem. Notice, that the (dimensionless) heat conductivity $\frac{15}{4}KnT$ depends on the temperature. We prescribe $T\left(x=0\right) = \vartheta_0$ and $T\left(x=1\right) = \vartheta_L$ to obtain

$$T = \sqrt{\vartheta_0^2 + \left(\vartheta_L^2 - \vartheta_0^2\right)x}, \qquad q = -\frac{15}{8}K_n\left(\vartheta_L^2 - \vartheta_0^2\right)\ . \tag{9}$$

The temperatures ϑ_0, ϑ_L are not the temperatures of the walls, but the temperatures of the gas directly at the walls and differ from the wall temperatures $T_0 = 1, T_L$. Indeed, in the case of large Knudsen numbers one has to consider the jump of the temperature at a wall which follows from the boundary conditions for the phase density [6, 27]. We consider diffusive boundary conditions for the gas at a boundary at rest, where the particles

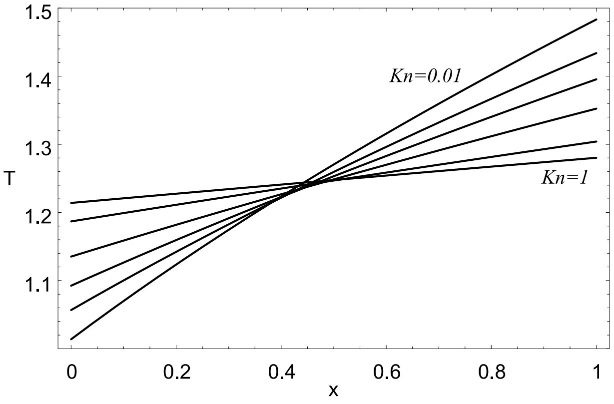

FIG. 1. *Temperature according to Fourier's law with temperature jumps for Knudsen numbers* $Kn = 0.01, 0.05, 0.1, 0.2, 0.5, 1$*; wall temperatures* $T_0 = 1, T_L = 1.5$.

leave the walls in Maxwellian distributions determined by the temperatures of the walls. Written in dimensionless form the temperature jumps at $x = 0, x = 1$ are given by [24, 28]

$$\frac{1-\vartheta_0}{\vartheta_0} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{q}{\sqrt{\vartheta_0}} \quad , \qquad \frac{T_L-\vartheta_L}{\vartheta_L} = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{q}{\sqrt{\vartheta_L}} \ , \tag{10}$$

ϑ_0 and ϑ_L follow from (10) with $(9)_2$. Fig. 1 shows the temperature for various Knudsen numbers (wall temperatures $T_0 = 1, T_L = 1.5$). The jumps increase with increasing Knudsen number.

Although there are jumps, no boundary layers are present in the 13 moment case. Indeed, by (9) the temperature curve is independent of the Knudsen number. This is not so when more moments are taken into account, as will be seen when we study a case with 26 moments. The equations follow from the procedures described in [28] as

$$\begin{aligned}
\frac{\partial q}{\partial x} &= 0 \\
\frac{\partial}{\partial x}\left(\frac{8}{15}q + \varphi\right) &= -\frac{1}{Kn}\frac{(1-\sigma)\,\sigma}{T} \\
\frac{\partial}{\partial x}\left(\frac{1}{3}\chi + \psi\right) &= -\frac{4}{3Kn}\frac{(1-\sigma)\,q}{T} \\
\frac{\partial}{\partial x}\left(\frac{9}{35}\psi\right) &= -\frac{3}{2Kn}\frac{(1-\sigma)\,\varphi}{T} \\
\frac{\partial}{\partial x}\left(28qT\right) &= -\frac{2}{3Kn}\frac{(1-\sigma)}{T}\left(\chi - 15T(1-\sigma)\right) \\
\frac{\partial}{\partial x}\left(\frac{112}{15}qT + 9T\varphi\right) &= -\frac{7}{6Kn}\frac{(1-\sigma)}{T}\left(\psi - T\sigma\right) \ .
\end{aligned} \tag{11}$$

Here, we have introduced the abbreviations φ, χ, ψ for some components of higher order moments. Below we shall present some numerical results for the full non-linear problem. For now, however, we are interested only in very small deviations of a global equilibrium, where $T_E = 1, \sigma_E = 0, q_E = 0, \varphi_E = 0, \chi_E = 15, \psi_E = 0$. Considering only first order terms in deviations from this equilibrium state, we find

$$\begin{aligned} q &= const.\ , & \frac{\partial \varphi}{\partial x} &= -\frac{1}{Kn}\sigma\ , \\ \frac{\partial}{\partial x}\left(\frac{1}{3}\chi + \psi\right) &= -\frac{4}{3Kn}q\ , & \frac{\partial}{\partial x}\left(\frac{9}{35}\psi\right) &= -\frac{3}{2Kn}\varphi\ , \\ 0 &= -\frac{2}{3Kn}\left(\chi - 15\left(T - \sigma\right)\right)\ , & 9\frac{\partial \varphi}{\partial x} &= -\frac{7}{6Kn}\left(\psi - \sigma\right) \end{aligned}$$

χ and ψ can easily be computed as

$$\chi = 15\left(T - \sigma\right)\ , \qquad \psi = \frac{61}{7}\sigma$$

and the remaining equations reduce to

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{Kn}\sigma\ , \quad \frac{\partial}{\partial x}\left(T + \frac{26}{35}\sigma\right) = -\frac{4}{15}\frac{1}{Kn}q\ , \quad \frac{366}{245}\frac{\partial \sigma}{\partial x} = -\frac{1}{Kn}\varphi\ .$$

Since the heat flux is constant, we can integrate the second equation to

$$T = K - \frac{4}{15}\frac{qx}{Kn} - \frac{26}{35}\sigma \tag{12}$$

while the two other equations give for the stress

$$\sigma = A\cosh\sqrt{\frac{245}{366}}\frac{x - 0.5}{Kn} + B\sinh\sqrt{\frac{245}{366}}\frac{x - 0.5}{Kn}\ . \tag{13}$$

A, B, K and q are the four constants of integration and four boundary conditions are required for their determination. Thus, the two boundary conditions for the wall temperatures - or the temperature jumps, respectively, - are not sufficient for a complete solution of the problem. In [28] we give a detailed discussion how the boundary conditions for the phase density can be implemented in a numerical scheme, the so-called kinetic schemes. Also, see [26, 27, 20] for another method of finding additional boundary conditions.

The boundary conditions are not necessary for a general discussion of the result: The first two terms in (12) give the solution of the linearized Fourier law, i.e. a straight temperature curve. The third term, $-\frac{26}{35}\sigma$, gives the deviation from the Fourier law, due to the influence of the higher moments. Our numerical results in [28] suggest that the leading term in the

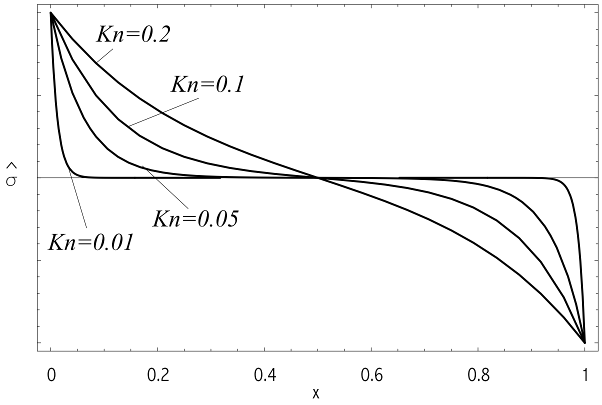

FIG. 2. *The function $\hat{\sigma}$ for various Knudsen numbers Kn (arbitrary units).*

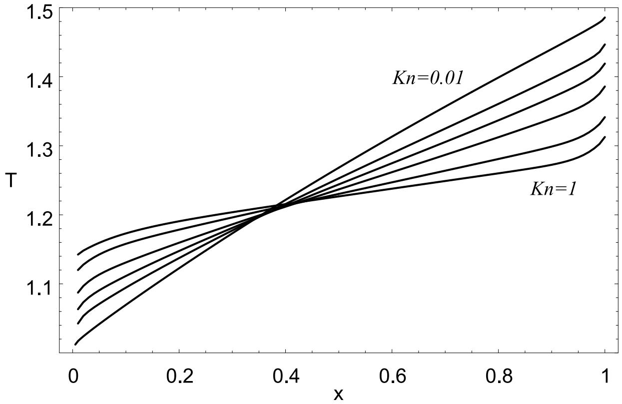

FIG. 3. *Temperature curve for Knudsen numbers $Kn = 0.01, 0.05, 0.1, 0.2, 0.5, 1$, wall temperatures $T_0 = 1, T_L = 1.5$.*

stress is given by $\hat{\sigma} = -B\sinh\sqrt{\frac{245}{366}}\frac{x-0.5}{Kn}$ and Fig. 2 shows this function for various Knudsen numbers (arbitrary units, normalized). It can be seen that this deviation has the form of a boundary layer indeed. Evidently, the thickness of the boundary layer increases with the Knudsen number.

We emphasize again that in the case of one-dimensional stationary heat transfer the Burnett equations for Maxwell molecules (6, 7) reduce to the Fourier law (8) and therefore cannot describe a linear boundary layer.

Fig. 3 shows the temperature for a variety of Knudsen numbers between $Kn = 0.01$ and $Kn = 1$, again for the wall temperatures $T_0 = 1, T_L = 1.5$, computed numerically in [28]. All curves were calculated with sufficiently large moment numbers, so that the results did not change when

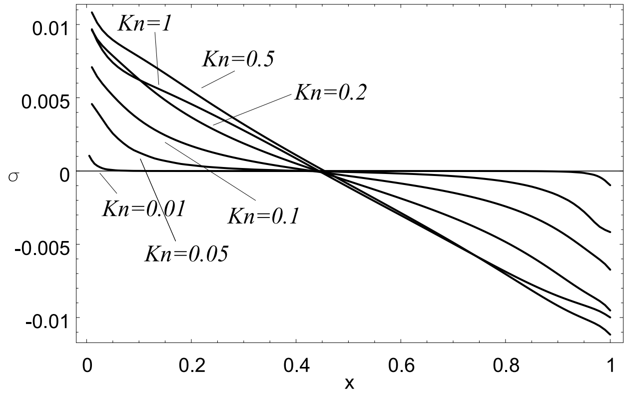

FIG. 4. *Anisotropic stresses for various Kn.*

more moments were added. The figure must be compared with Fig. 1, which shows the results for the Fourier case. Evidently, the moment solution gives smaller jumps, and adds marked boundary layers.

The growth of the boundary layer with increasing Kn can best be seen from the curves of the anisotropic stresses in Fig. 4. With $Kn = 0.01$ the stress indeed differs from zero only in a small layer at the walls. With increasing Kn the boundary layer expands more and more into the gas. Already at $Kn = 0.1$ both boundary layers meet, and the rarefied gas effects dominate the anisotropic stresses σ.

Acknowledgment. This paper continues discussions with Marshall Slemrod (Madison) and Ramesh Agarwal (Wichita) at the Institute for Mathematics and its Applications (Minneapolis) during its program on "Reactive Flow and Transport Phenomena" in Spring 2000. I wish to thank the IMA for their hospitality and support.

REFERENCES

[1] J.D. AU, *Nichtlineare Probleme und Lösungen in der Erweiterten Thermodynamik.* Dissertation, Technical University Berlin 2000.

[2] R. BALAKRISHNAN, *Entropy consistent formulation and numerical simulation of the BGK-Burnett equations for hypersonic flows in the continuum-transition regime.* Dissertation, Wichita State University 1999.

[3] G. BIRD, *Molecular gas dynamics and the direct simulation of gas flows.* Clarendon Press, Oxford 1994.

[4] A.V. BOBYLEV, *The Chapman-Enskog and Grad methods for solving the Boltzmann equation.* Sov. Phys. Dokl. **27**: 29–31 (1982).

[5] C. CERCIGNANI, *Theory and application of the Boltzmann Equation.* Scottish Academic Press, Edinburgh 1975.

[6] S. CHAPMAN AND T.G. COWLING, *The mathematical Theory of Non-Uniform Gases.* Cambridge University Press 1970.

[7] K.A. Comeaux, D.R. Chapman, and R.W. MacCormack, *An analysis of the Burnett equations based on the second law of thermodynamics.* AIAA Paper 95-0415, 1995.

[8] H. Grad, *On the Kinetic Theory of Rarefied Gases.* Comm. Pure Appl. Math. **2**: 325 (1949).

[9] J.H. Ferziger and H.G. Kaper, *Mathematical theory of transport processes in gases.* North-Holland, Amsterdam 1972.

[10] S. Jin and M. Slemrod, *Regularization of the Burnett equations via relaxation.* Preprint, University of Wisconsin-Madison (2000).

[11] M.N. Kogan, *Rarefied Gas Dynamics.* Plenum Press, New York 1969.

[12] R.L. Liboff, *The Theory of Kinetic Equations.* Wiley and Sons, New York 1969.

[13] D.W. Mackowski, D.H. Papadopoulos, and D.E. Rosner, *Comparison of Burnett and DSMC predictions of pressure distributions and normal stress in one-dimensional, strongly nonisothermal gas.* Phys. Fluids **11**: 2108–2116 (1999).

[14] I. Müller, *On the frame dependence of stress and heat flux.* Arch. Rational Mech. Anal. **45**: 241–250 (1972).

[15] I. Müller, *Thermodynamics*, Pitman, Boston 1985.

[16] I. Müller and T. Ruggeri, *Rational Extended Thermodynamics.* Springer, New York 1998 (Springer Tracts in Natural Philosophy Vol. **37**).

[17] T. Ohwada, *Heat flow and temperature and density distributions in a rarefied gas between parallel plates with different temperatures. Finite difference analysis of the nonlinear Boltzmann equation for hard sphere molecules.* Phys. Fluids **8**: 2153–2160 (1996).

[18] S. Reinecke and G.M. Kremer, *Method of Moments of Grad.* Phys. Rev. A **42**: 815–820 (1990).

[19] S. Reinecke and G.M. Kremer, *Burnett's equations from a (13+9N)-field theory.* Cont Mech. Thermodyn. **8**: 121–130 (1996).

[20] D. Reitebuch and W. Weiss, *Application of High Moment Theory to the Plane Couette Flow.* Cont. Mech. Thermodyn. **11**: 227 (1999).

[21] P. Rosenau, *Extending hydrodynamics via the regularization of the Chapman-Enskog solution.* Phys. Rev. A **40**: 7193–7196 (1989).

[22] M. Slemrod, *Constitutive relations for a monatomic gas based on a generalized approximation to the sum of the Chapman-Enskog expansion.* Arch. Rational Mech. Anal. **150**: 1–22 (1999).

[23] H. Struchtrup, *An Extended Moment Method in Radiative Transfer: The Matrices of Mean Absorption and Scattering Coefficients.* Ann. Phys. **257**: 111–135 (1997).

[24] H. Struchtrup, *Kinetic schemes and boundary conditions for moment equations.* ZAMP **51**: 346–365 (2000).

[25] H. Struchtrup, *Positive and negative entropy productions and phase densities for approximate solutions of the Boltzmann equation.* J. Thermophys. Heat Trans., **15**(3): 372–373 (2001).

[26] H. Struchtrup and W. Weiss, *Maximum of the local entropy production becomes minimal in stationary processes.* Phys. Rev. Lett. **80**: 5048–5051 (1998).

[27] H. Struchtrup and W. Weiss, *Temperature jumps and velocity slip in the moment method.* Cont. Mech. Thermodyn. **12**: 1–18 (2000).

[28] H. Struchtrup, *Heat Transfer in the Transition Regime: Solution of Boundary Value Problems for Grad's Moment Equations via Kinetic Schemes.* Phys. Rev. E **65**, 041204 (2002).

[29] E. Ikenberry and C. Truesdell, *On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory I.* J. of Rat. Mech. Anal. **5**: 1–54 (1956).

[30] C. Truesdell and R.G. Muncaster, *Fundamentals of Maxwell's kinetic theory of a simple monatomic gas.* Academic Press, New York, 1980.

[31] F.J. Uribe, R.M. Velasco and L.S. García-Colín, *Burnett description of strong shock waves.* Phys. Rev. Lett. **81**: 2044–2047 (1998).

[32] F.J. Uribe and A.L. Garcia, *Burnett description of plane Poiseuille flow.* Phys. Rev. E **60**: 4063–4078 (1999).

[33] W. Weiss, *Continuous shock structure in extended Thermodynamics.* Phys. Rev. E **52**: 5760 (1995).

[34] X. Zhong, R.W. MacCormack, and D.R. Chapman, *Stabilization of the Burnett equations and application to hypersonic flows.* AIAA Journal **31**: 1036–1043 (1993).