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Projected moments in relativistic kinetic theory

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Abstract

In this paper a new set of moment equations in relativistic kinetic theory is presented. The moments under consideration are the projections of particle 4-flux and energy momentum tensor with respect to the Eckart velocity or the Landau–Lifshitz velocity, alternatively. The moment equations follow from integrations of the relativistic Boltzmann equation in which the interactions of the particles are described by the relativistic BGK model for reasons of simplicity. The projected moment formalism is extended to an arbitrary number of moments and moment equations and it is shown that the non-relativistic limit of moments and moments equations leads to the so-called central moments of non-relativistic theory. The moment equations may be closed by means of the entropy maximum principle. After this method has been outlined, the closure is performed for the case of 14 moments, i.e. the projections of particle 4-flux and energy momentum tensor. Moreover local thermal equilibrium is considered where the projected moment formalism is used for the derivation of the relativistic Navier–Stokes and Fourier laws. Different choices of moment equations for this task are compared and it is shown that the proper choice of moment equations depends on the interaction term in the relativistic Boltzmann equation.

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1. Introduction

The objective of relativistic thermodynamics is the calculation of particle 4-flux N^A and energy momentum tensor T^{AB}. The conservation laws for particle number and energy momentum,

$$N^A{}_{,A} = 0\,, \tag{1a}$$

$$T^{AB}{}_{,B} = 0\,, \tag{1b}$$

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provide five equations for the determination of the 14 elements of N^A and T^{AB}. Thus, nine additional equations are required in order to have a closed system. The usual procedure in relativistic thermodynamics is to assume an additional equation which reads

$$A^{ABC}{}_{,C} = P^{AB} . \tag{2}$$

Here A^{ABC} is a completely symmetric 4-tensor with

$$A^{AB}{}_{B} = m^2 c^2 N^A , \tag{3}$$

where m is the particle mass and c is the speed of light. The trace of the production tensor P^{AB} must vanish due to Eqs. (1a) and (3),

$$P^{A}{}_{A} = 0 . \tag{4}$$

The form of Eq. (2) as well as the properties of A^{ABC} and P^{AB} are motivated from the relativistic kinetic theory and will be derived later in this paper. Eq. (2) gives nine equations since its trace is equal to Eq. (1a). But still the system is not closed since a priori A^{ABC} is not related to N^A and T^{AB}. The closure requires constitutive equations which relate the new quantity A^{ABC} to N^A and T^{AB}. For the various methods of closing we refer the reader to the literature [1–9]

In this paper we present an alternative approach which involves moment equations for the projections of N^A and T^{AB} with respect to an observer velocity – quantities with physical meaning – rather than a moment equation for the abstract quantity A^{ABC}. We shall use the formalism of projected moments which was introduced in relativistic kinetic theory of radiation by Thorne [10], see also [11–13].

It will become clear in the sequel that the projected moment formalism distinguishes one observer frame which has physical significance. The variables under consideration are the relativistic generalizations of the moments measured in this observer frame. In relativistic thermodynamics one has the choice between the Eckart frame, where the particle flux vanishes and the Landau–Lifshitz frame, where the energy flux vanishes – both frames will be considered in this paper.

For reasons of generality, we shall develop the theory for an arbitrary number of moments and perform the restriction to the projections of N^A and T^{AB} later. In particular, we shall emphasize the strong relationship between the use of projected moments in relativistic kinetic theory and the use of central moments in classical kinetic theory. Since the choice of the moment equations is not affected by fine details of the particle interaction processes, it is sufficient in the present context to rely on the relativistic Boltzmann equation with the BGK-model for the collisions. Most of the paper deals with the relativistic BGK model of Anderson & Witting [14], but we shall also use the model of Marle [4] for discussions.

The paper is organized as follows: In Section 2 we introduce the basic quantities and equations of relativistic kinetic theory. In particular, we define the phase density and some of its moments – particle 4-flux and energy momentum tensor – and present the relativistic Boltzmann equation with BGK model. Section 3 contains the definition of projected and unprojected moments and the derivation of the moment equations from the Boltzmann equation. Here we also discuss the non-relativistic limit of the moments and show that the projected moments reduce to the central moments of classical kinetic theory. The set of moment equations is not closed and Section 4 deals with the closure by means of the entropy maximum principle [15,8]. The principle is formulated for an arbitrary number of moments but we perform the explicit calculations only for the case of 14 moments, particle 4-flux and energy momentum tensor. The resulting system of field equations differs from the usual Eqs. (1) and (2) and Section 5 discusses the differences in the case of local thermal equilibrium. We use a Maxwellian iteration procedure to obtain the relativistic Navier–Stokes and Fourier laws and show that the results depend on the choice of moment equations. Moreover we compare the results with those which follow from the Chapman–Enskog method and give reasons that the proper choice of moment equations depend on the details of the BGK model under consideration.

2. Phase density, moments and relativistic Boltzmann equation

2.1. Basics

We restrict ourselves to flat space time with space-time variables

$$x^A = \{ct, x^i\}^A, \qquad A = 0,1,2,3\,,$$

where t denotes time and x^i denotes the space variable, the Lorentz metric is given by

$$\eta_{AB} = \begin{Bmatrix} 1 & 0 \\ 0 & -\delta_{ij} \end{Bmatrix}_{AB}.$$

The particle 4-momentum is denoted by

$$p^A = \{p^0, p^i\}\,, \tag{5}$$

where cp^0 is the energy of the particle and p^i is the 3-momentum. The absolute value of p^A is constant,

$$p^A p_A = p^A p^B \eta_{AB} = (p^0)^2 - p^i p^i = m^2 c^2\,, \tag{6}$$

and the 3-velocity of the particle is given by

$$\xi^i = c\frac{p^i}{p^0}\,. \tag{7}$$

Hence, follows a relation between particle energy and velocity which we shall need later, viz.

$$p^0 = \frac{mc}{\sqrt{1-(\xi^2/c^2)}}\,. \tag{8}$$

Observer 4-velocities are denoted by U^A with

$$U^A U_A = c^2 \tag{9}$$

and in the observer's Lorentz rest frame (LRF) we have

$$U^A_{|LR} = \{c, 0, 0, 0\}^A . \tag{10}$$

We shall also make use of the spatial projector

$$\Delta^{AB} = \eta^{AB} - \frac{1}{c^2} U^A U^B , \tag{11}$$

which has the following properties:

$$\Delta^{AB} U_B = 0, \qquad \Delta^{AB} \Delta_{BC} = \Delta^A{}_C, \qquad \Delta^A{}_A = 3 . \tag{12}$$

2.2. Phase density

We consider monatomic ideal gases whose state is completely described, if the phase density $f(x^A, p^i)$ is known. The phase density is defined such that

$$f(x^A, p^i)\, dX\, dP ,$$

gives the number of atoms in the element $dX\, dP = p^0 d^3x\, dP$ [4,1]. Due to this definition f is an invariant scalar. dP denotes the invariant momentum space element

$$dP = \frac{d^3 p}{p^0} . \tag{13}$$

The particle 4-flux and the energy momentum tensor are moments of the phase density, given by

$$N_A = c \int p^A f\, dP, \qquad T^{AB} = c \int p^A p^B f\, dP . \tag{14}$$

The entropy 4-vector is given by [9]

$$S^A = -kc \int p^A f \ln \frac{f}{y}\, dP , \tag{15}$$

where $y = const$ is the volume of a phase-space cell and k denotes Boltzmann's constant. This definition follows from considerations on the relativistic Boltzmann equation. It should be mentioned that we need the definition of entropy (15) as an additional input in our considerations since we do not consider the Boltzmann interaction term but only the BGK relaxation model, see below.

2.3. Decomposition of p^A

We consider an observer with 4-velocity U^A and decompose the 4-momentum into one part parallel to U^A and one part perpendicular to U^A [10,13]

$$p^A = p^0_{|LR} \frac{1}{c} U^A + R^A \quad \text{with } R^A U_A = 0 . \tag{16}$$

Because of

$$p^A_{|LR} = \{p^0_{|LR}, R^i_{|LR}\} \tag{17}$$

$cp^0_{|LR}$ is the particle energy in LRF and R^A is the covariant generalization of the 3-momentum in LRF. We have

$$p^0_{|LR} = p^A \frac{1}{c} U_A \tag{18a}$$

and

$$R^A = p^B \Delta^A{}_B \quad \text{with } R^A R_A = m^2c^2 - (p^0_{|LR})^2. \tag{18b}$$

By means of Eq. (16) we may decompose the moments (14) according to

$$\begin{aligned} N^A &= nU^A + J^A\,, \\ T^{AB} &= e\frac{1}{c^2}U^A U^B + F^A \frac{1}{c} U^B + F^B \frac{1}{c} U^A + P^{AB}\,. \end{aligned} \tag{19}$$

The newly introduced quantities are defined as

$$\begin{aligned} n &= \int p^0_{|LR} f\, dP, \qquad J^A = c\int R^A f\, dP\,, \\ e &= c\int (p^0_{|LR})^2 f\, dP, \qquad F^A = c\int p^0_{|LR} R^A f\, dP, \qquad P^{AB} = c\int R^A R^B f\, dP \end{aligned} \tag{20}$$

with

$$J^A U_A = 0, \qquad F^A U_A = 0, \qquad P^{AB} U_B = 0\,. \tag{21}$$

These quantities are covariant generalizations of moments measured in LRF: n is the particle number density, J^A is the particle flux, e is the energy density, $(1/c)F^A$ is the momentum density and P^{AB} is the pressure tensor. The trace of the pressure tensor is related to the pressure p by

$$-3p = P^A{}_A\,. \tag{22}$$

The vectors and tensors under consideration have no parts in the direction of U^A (21): they are projected in the plane perpendicular to U^A. That is why we speak of projected moments.

2.4. Eckart and Landau–Lifshitz velocities

The decompositions (16) and (19) may be performed with respect to any observer velocity U^A, but there are two choices with physical meaning: We may choose the observer frame such that the particle flux vanishes. The velocity of this frame is called Eckart velocity W^A and we have [9]

$$\begin{aligned} N^A &= n_E W^A\,, \\ T^{AB} &= e_E\frac{1}{c^2}W^A W^B + F_E^A \frac{1}{c} W^B + F_E^B \frac{1}{c} W^A + P_E^{AB}\,. \end{aligned} \tag{23}$$

The Landau–Lifshitz velocity V^A is defined such that the momentum density vanishes and we have [9]

$$N^A = n_{LL} V^A + J^A_{LL} \,, \qquad T^{AB} = e_{LL} \frac{1}{c^2} V^A V^B + P^{AB}_{LL} \,. \tag{24}$$

Almost all decompositions and projections in this paper are meant to be decompositions and projections with respect to one of these two velocities.

2.5. Equilibrium phase density

In this section we determine the equilibrium phase density $f_{|E}$ which we need as input for the relativistic BGK model. We employ the entropy maximum principle which we will use later in this paper also to determine non-equilibrium phase densities.

We know: (i) The equilibrium state of the gas under consideration is determined completely by the conserved quantities number density n, energy density e, and momentum density F^A. (ii) The entropy density $\eta = (1/c^2) S^A U_A$ reaches its maximum value in equilibrium. Thus, the equilibrium phase density is *the* phase density which maximizes

$$\eta = \frac{1}{c^2} S^A U_A = -k \int p^0_{|LR} f \ln \frac{f}{y} \, dP \,, \tag{25}$$

under the constraints of prescribed values for n, e and F^A. Note that U^A may be any observer velocity in the present context. We take care of the constraints by Lagrange multipliers and maximize

$$-k \int p^0_{|LR} f \ln \frac{f}{y} \, dP - \lambda_n \left(\int p^0_{|LR} f \, dP - n \right)$$
$$-\lambda_e \left(\int (p^0_{|LR})^2 f \, dP - \frac{e}{c} \right) - \lambda_A \left(\int p^0_{|LR} R^A f \, dP - \frac{1}{c} F^A \right)$$

without constraints. We obtain

$$f_{|E} = y \exp\left\{ -1 - \frac{1}{k} [\lambda_n + \lambda_e p^0_{|LR} + \lambda_A R^A] \right\} , \tag{26}$$

where the λ's are functions of n, e and F^A. In particular we must have $\lambda_A = \gamma(n, e, F^B) F_A$ so that $\lambda_A = 0$ in the Landau–Lifshitz frame, $U^A = V^A$. Therefore, we may write

$$f_{|E} = \mathscr{A} \exp\{-\beta c p^0_{|LL}\} \quad \text{with} \;\; p^0_{|LL} = p^A \frac{1}{c} V_A \,. \tag{27}$$

$\mathscr{A}$ and β are abbreviations which replace λ_n and λ_e; they follow from the constraints

$$n_{LL} = \int p^0_{|LL} f_{|E} \, dP, \quad e_{LL} = c \int (p^0_{|LR})^2 f_{|E} \, dP \tag{28}$$

as [7]

$$\beta=\frac{1}{kT}\,,\qquad \mathscr{A}=\frac{n_{LL}}{4\pi(mc/z)^3\int_z^\infty e^{-\gamma}\sqrt{\gamma^2-z^2}\gamma\,d\gamma}\quad\text{with } z=\frac{mc^2}{kT}\,. \tag{29}$$

T is the thermodynamic temperature which may be identified via the Gibbs equation [7]. The equilibrium values of the moments follow from Eq. (14) with Eq. (27) as

$$N^A_{|E}=n_{LL}V^A\,,\qquad T^{AB}_{|E}=e_{LL}\frac{1}{c^2}V^AV^B-p_{LL}\,\Delta^{AB}_{LL}\,, \tag{30}$$

where

$$\Delta^{AB}_{LL}=\eta^{AB}-\frac{1}{c^2}V^AV^B\,.$$

From Eqs. (30) and (19) we conclude that the equilibrium particle flux $J^A_{|E}$ vanishes in the Landau–Lifshitz frame. This means – according to Eqs. (23) and (24) – that there is no difference between Eckart frame and Landau–Lifshitz frame in equilibrium. Energy density e_{LL} and pressure p_{LL} in Eq. (30) are related to T and n_{LL} by

$$p_{LL}=n_{LL}kT\,,\qquad e_{LL}=n_{LL}\frac{mc^2}{z}\frac{\int_z^\infty e^{-\gamma}\sqrt{\tau^2-z^2}\gamma^2\,d\gamma}{\int_z^\infty e^{-\gamma}\sqrt{\gamma^2-z^2}\gamma\,d\gamma}\,. \tag{31}$$

The calculation of the relativistic equilibrium phase density is due to Jüttner [16].

2.6. Relativistic Boltzmann equation with BGK model

We follow Anderson and Witting [14] and write the relativistic Boltzmann equation with BGK model as

$$p^Af_{,A}=-\frac{1}{c^2\tau}p^AV_A(f-f_{|E})\,. \tag{32}$$

Here τ is a relaxation time which may be interpreted as the mean collision-free time of a particle, measured in the Landau–Lifshitz frame. See Appendix A for a short motivation of Eq. (32).

Integration of the Boltzmann equation gives the balance of particle number density,

$$N^A{}_{,A}=-\frac{1}{c^2\tau}V_A(N^A-N^A_{|E})=-\frac{1}{c^2\tau}V_A(n_{LL}V^A+J^A_{LL}-n_{LL}V^A)=0\,, \tag{33}$$

where we have used Eqs. (24) and (30) and $V_A J_{LL}^A = 0$. Multiplication of Eq. (32) with p^B and integration gives the energy–momentum equation,

$$T^{AB}{}_{,B} = -\frac{1}{c^2\tau} V_B (T^{AB} - T^{AB}_{|E})$$
$$= -\frac{1}{c^2\tau} V_B \left(e_{LL}\frac{1}{c^2} V^A V^B + P^{AB}_{LL} - e_{LL}\frac{1}{c^2} V^A V^B - p_{LL}\,\Delta^{AB}_{LL} \right) = 0\,, \tag{34}$$

where we have used Eqs. (24) and (30) and $P^{AB}_{LL} V_B = 0, \Delta^{AB}_{LL} V_B = 0$. Thus, the relativistic BGK model guaranties the conservation of particle number, energy and momentum.

It should be noted that Marle [4] presents an alternative relativistic BGK equation, viz.

$$p^A f_{,A} = -\frac{m}{\tau}(f - f_M)\,. \tag{35}$$

Here f_M is a phase density of the form

$$f_M = \mathscr{B} \exp\{-\gamma c p^0_{|LL}\} \tag{36}$$

where $\mathscr{B}$ and γ must be determined from the conservation requirements

$$N^A{}_{,A} = -\frac{m}{\tau}\int (f - f_M)\, dP = 0, \qquad T^{AB}{}_{,B} = -\frac{m}{\tau}\int p^A (f - f_M)\, dP = 0\,. \tag{37}$$

Thus, f_M is not the equilibrium phase density. We prefer the Anderson–Witting version of the BGK model, because it has the same features as the classical BGK model [17], i.e. $f_{|E}$ is the equilibrium distribution and one finds the same moments of the phase density on the right and left hand side of the moment equations, see Eqs. (33) and (34) and Eqs. (46) and (53) below. Moreover, it gives a proper interpretation of the relaxation time: τ is the mean collision-free time of an atom, measured in the Landau–Lifshitz frame. We will come back to Marle's BGK model at the end of Section 5.

2.7. Balance of entropy, *H*-theorem

We end this section with the proof of the H-theorem for the relativistic BGK model. The divergence of the entropy 4-vector (15) reads

$$S^A{}_{,A} = -kc \int p^A \left(\ln\frac{f}{y} + 1 \right) f_{,A} dP$$
$$= \frac{k}{c\tau} V_A \int p^A \left(\ln\frac{f}{y} + 1 \right) (f - f_{|E})\, dP = \sigma\,, \tag{38}$$

where σ is the density of entropy production. Due the conservation laws for particle number and energy momentum we have

$$\frac{k}{c\tau} V_A \int p^A \left(\ln\frac{f_{|E}}{y} + 1 \right) (f - f_{|E})\, dP = 0\,. \tag{39}$$

We subtract this from Eq. (38) and obtain

$$S^{A}{}_{,A} = \frac{k}{c\tau} V_A \int p^A \ln \frac{f}{f_{|E}} (f - f_{|E})\, dP = \sigma \geqslant 0\,; \tag{40}$$

σ is always positive – i.e. the H-theorem is valid.

3. Moments and moment equations

3.1. Unprojected moments

We define unprojected moments of the phase density by [11,10,13]

$$A_r^{A_1\cdots A_N} = c \int (p^0_{|LR})^{r+1-N} p^{A_1} \cdots p^{A_N} f\, dP\,. \tag{41}$$

Because of $p^0_{|LR} = (1/c) p^A U_A$ this definition distinguishes the frame which moves with velocity U^A. The unprojected moments are fully symmetric 4-tensors with the following properties:

$$A_r^{A_1\cdots A_N} \frac{1}{c} U_{A_N} = A_r^{A_1\cdots A_{N-1}}\,, \tag{42a}$$

$$A_r^{A_1\cdots A_{N-2} A_{N-1}}{}_{A_{N-1}} = m^2 c^2 A_{r-2}^{A_1\cdots A_{N-2}}\,. \tag{42b}$$

The moments with $r = N-1$ are the usual moments of relativistic kinetic theory and are independent of U^A,

$$A_{N-1}^{A_1\cdots A_N} = c \int p^{A_1} \cdots p^{A_N} f\, dP\,. \tag{43}$$

Particle 4-flux and energy momentum tensor are unprojected moments by

$$N^A = A_0^A, \qquad T^{AB} = A_1^{AB}. \tag{44}$$

Because of Eq. (42a) they are contained in the moments $A_0^{A_1\cdots A_N}$ ($N \geqslant 1$) and $A_1^{A_1\cdots A_N}$ ($N \geqslant 2$), respectively.

Multiplication of the relativistic Boltzmann equation (32) with $c(p^0_{|LR})^{r-N} p^{A_1} \cdots p^{A_N}$ $1/p^0$ and subsequent integration over d^3p yields the moment equations for the unprojected moments. With use of the identities

$$p^A{}_{,B} = 0, \qquad (p^0_{|LR})_{,A} = \frac{1}{c} p^B U_{B,A}\,, \tag{45}$$

we obtain

$$A_r^{A_1\cdots A_N B}{}_{,B} + (N-r) A_r^{A_1\cdots A_N BC} \frac{1}{c} U_{C,B} = -\frac{1}{c\tau} (A_r^{A_1\cdots A_N B} - A_r^{A_1\cdots A_N B}{}_{|E}) \frac{1}{c} V_B\,, \tag{46}$$

where we have defined the equilibrium values of the moments by

$$A_r^{A_1\cdots A_N B}{}_{|E} = c \int (p^0_{|LR})^{r-N} p^{A_1} \cdots p^{A_N} p^B f_{|E}\, dP\,. \tag{47}$$

3.2. Projected moments

We define projected moments by

$$M_r^{A_1\cdots A_n} = c\int (p^0_{|LR})^{r+1-n} R^{A_1}\cdots R^{A_n} f\, dP\,. \tag{48}$$

Comparison with Eq. (20) shows the physical interpretation of some of the projected moments

$$\begin{aligned} &M_0 = cn, \qquad M_0^A = J^A\,,\\ &M_1 = e, \qquad M_1^A = F^A, \qquad M_1^{AB} = P^{AB}\,. \end{aligned} \tag{49}$$

The relation between projected and unprojected moments follows by means of Eqs. (18a) and (18b) as

$$M_r^{A_1\cdots A_n} = \Delta^{A_1}{}_{B_1}\cdots \Delta^{A_n}{}_{B_n} A_r^{B_1\cdots B_n}\,. \tag{50}$$

By means of Eq. (16) the moments (41) may be decomposed as

$$A_r^{A_1A_2\cdots A_N} = \sum_{k=0}^{N}\binom{N}{k}\frac{1}{c^{N-k}} M_r^{(A_1\cdots A_k} U^{A_{k+1}}\cdots U^{A_N)}\,, \tag{51}$$

where the brackets indicate symmetrization. Due to Eqs. (16), (18a) and (18b) the projected moments have the properties

$$M_r^{A_1\cdots A_n}\frac{1}{c}U_{A_n} = 0, \quad M_r^{A_1\cdots A_{n-2}A_{n-1}}{}_{A_{n-1}} = m^2c^2 M_{r-2}^{A_1\cdots A_{n-2}} - M_r^{A_1\cdots A_{n-2}}\,. \tag{52}$$

Multiplication of Eq. (46) for $N=n$ with $\Delta_{A_1}{}^{B_1}\cdots\Delta_{A_n}{}^{B_n}$ yields after some rearrangement the moment equations for projected moments,

$$\begin{aligned} &DM_r^{B_1\cdots B_n} + \nabla_C M_r^{B_1\cdots B_n C}\\ &\quad+\frac{1}{c}DU_D\left\{(n-r-1)M_r^{B_1\cdots B_n D} + n\eta^{D(B_1}M_r^{B_2\cdots B_n)} + n\frac{1}{c}U^{(B_1}M_r^{B_2\cdots B_n)D}\right\}\\ &\quad+\frac{1}{c}\nabla_C U_D\left\{(n-r)M_r^{B_1\cdots B_n CD} + M_r^{B_1\cdots B_n}\eta^{CD}\right.\\ &\quad\left.+n\eta^{D(B_1}M_r^{B_2\cdots B_n)C} + n\frac{1}{c}U^{(B_1}M_r^{B_2\cdots B_n)CD}\right\}\\ &\quad= -\frac{1}{c\tau}\left\{(M_r^{B_1\cdots B_n} - M_r^{B_1\ldots B_n}{}_{|E})\frac{1}{c^2}V_AU^A + (M_r^{B_1\cdots B_nA} - M_r^{B_1\ldots B_nA}{}_{|E})\frac{1}{c}V_A\right\}. \end{aligned} \tag{53}$$

The abbreviations D and ∇_C stand for the covariant generalizations of the partial derivatives in LRF with respect to time and space, respectively,

$$D\Psi = \frac{1}{c}U^C\Psi_{,C}, \qquad \nabla_C\Psi = \Delta_C^D\Psi_{,D} \qquad \text{so that } \Psi_{,C} = \frac{1}{c}U_C D\Psi + \nabla_C\Psi\,. \tag{54}$$

The moments of the equilibrium phase density which appear on the right-hand side of (53) are defined by

$$M_r^{B_1 \cdots B_n}|_E = c \int (p^0_{|LR})^{r+1-n} R^{B_1} \cdots R^{B_n} f_{|E}\, dP \tag{55}$$

and will be calculated in the next section.

3.3. Moments of the equilibrium phase density

3.3.1. Relations between decompositions

We proceed with the calculation of the equilibrium moments (55). The equilibrium phase density (27) is isotropic in the Landau–Lifshitz frame which differs, in general, from the Lorentz rest frame under consideration. Indeed, we have the decompositions

$$p^A = p^0_{|LR} \frac{1}{c} U^A + R^A, \qquad p^A = p^0_{|LL} \frac{1}{c} V^A + S^A, \tag{56}$$

where S^A is the 4-vector generalization of the 3-momentum in the Landau–Lifshitz frame. In order to calculate the integrals we need the following relations between $p^0_{|LR}, R^A$ and $p^0_{|LL}, S^A$:

$$\begin{aligned} p^0_{|LR} &= p^0_{|LL} \frac{1}{c^2} V^A U_A + \frac{1}{c} S^A U_A, \\ R^A &= p^0_{|LL} \left(\frac{1}{c} V^A - \frac{1}{c^2} V^B U_B \frac{1}{c} U^A \right) + S^A - \frac{1}{c^2} S^B U_B U^A. \end{aligned} \tag{57}$$

U^A is either the Eckart or the Landau–Lifshitz velocity. We assume that the difference between both is small which is true for processes not too far from equilibrium and write

$$V^A = U^A + w^A. \tag{58}$$

In the following, we shall neglect all second-order terms in w^A. We have

$$1 = \frac{1}{c^2} V^A V_A = \frac{1}{c^2} U^A U_A + \frac{2}{c^2} U^A w_A + \frac{1}{c^2} w^A w_A \simeq 1 + \frac{2}{c^2} U^A w_A \tag{59}$$

and conclude that

$$U^A w_A = 0 \tag{60}$$

and hence

$$\frac{1}{c^2} V^A U_A = 1$$

in this approximation. Thus, Eq. (57) reduce to

$$p^0_{|LR} = p^0_{|LL} - \frac{1}{c} S^A w_A, \qquad R^A = p^0_{|LL} \frac{1}{c} w^A + S^A + \frac{1}{c^2} S^B w_B U^A. \tag{61}$$

3.3.2. Equilibrium moments in the Landau–Lifshitz frame

We define the equilibrium moments with respect to the Landau–Lifshitz frame by

$$m_r^{B_1\cdots B_n} = c\int (p^0_{|LL})^{r+1-n} S^{B_1}\cdots S^{B_n} f_{|E}\, dP\,. \tag{62}$$

$f_{|E}$ is isotropic in the Landau–Lifshitz frame such that

$$\begin{aligned}
m_r^A &= 0\,,\\
m_r^{AB} &= \tfrac{1}{3} m_r{}^D{}_D \Delta_{LL}^{AB} = \tfrac{1}{3}(m^2c^2 m_{r-2} - m_r)\Delta_{LL}^{AB}\,,\\
m_r^{ABC} &= 0\,,\\
m_r^{ABCD} &= \tfrac{1}{15} m_r{}^F{}_F{}^G{}_G(\Delta_{LL}^{AB}\Delta_{LL}^{CD} + \Delta_{LL}^{AC}\Delta_{LL}^{BD} + \Delta_{LL}^{AD}\Delta_{LL}^{BC})\,,\\
m_r^{ABCDF} &= 0
\end{aligned} \tag{63}$$

and so on. Therefore, we need to calculate the scalar moments m_r only. Using $p^0 = \sqrt{m^2c^2 + p^2}$ (6), where $p^2 = p^i p^i$, the phase density (27) and (13) we may write

$$m_r = \mathscr{A} c (mc)^r \int_0^\infty \sqrt{1 + \frac{p^2}{m^2c^2}}^{\,r} \exp\left\{ -\frac{mc^2}{kT}\sqrt{1+\frac{p^2}{m^2c^2}} \right\} d^3p\,. \tag{64}$$

With the substitutions $mc^2/kT = z$, $\gamma = z\sqrt{1+(p/mc)^2}$ we obtain

$$m_r = (mc)^r m_0 \frac{I_r}{I_0}\,, \tag{65a}$$

where

$$I_r = \frac{1}{z^r}\int_z^\infty e^{-\gamma}\sqrt{\gamma^2 - z^2}\,\gamma^{r+1}\, d\gamma\,. \tag{65b}$$

The integrals $I_r(z)$ may be expressed by modified Bessel functions of the second kind $K_n(z)$ by means of the recurrence formula

$$\begin{aligned}
&(r-1)I_{r-3} = (r+2)I_r - z(I_r - I_{r-2})\\
&\quad \text{with } I_{-1} = z^2K_1,\ \ I_0 = z^2K_2,\ \ I_1 = 3zK_2 + z^2K_1\,.
\end{aligned} \tag{66}$$

It is not possible to calculate the integral $I_{-2}(z)$ from Eq. (66); therefore, I_{-2} must be determined numerically. For the derivatives of I_r with respect to z one finds

$$\frac{dI_r(z)}{dz} = \frac{r}{z}(I_{r-2} - I_r) - I_{r-1}\,. \tag{67}$$

3.3.3. Equilibrium moments in other frames

Now, it is easy to calculate the moments (55) in the limit of small velocity difference w^A in terms of the $m_r^{B_1\cdots B_n}$. In this paper we shall need only the moments up to rank

$n=4$; they read

$$
\begin{aligned}
&M_{r|E} = m_r\,,\\
&M_r^A{}_{|E} = \left(m_r - \frac{r}{3} m_r^D{}_D\right) \frac{1}{c} w^A\,,\\
&M_r^{AB}{}_{|E} = \tfrac{1}{3} m_r^D{}_D \Delta^{AB}\,,\\
&M_r^{ABC}{}_{|E} = \left(\frac{1}{3} m_{rD}^D - \frac{r-2}{15} m_r^F{}_F{}^G{}_G\right)\left(\Delta^{BC}\frac{1}{c}w^A + \Delta^{AC}\frac{1}{c}w^B + \Delta^{AB}\frac{1}{c}w^C\right),\\
&M_r^{ABCD}{}_{|E} = \tfrac{1}{15} m_r^F{}_F{}^G{}_G (\Delta^{AB}\Delta^{CD} + \Delta^{AC}\Delta^{BD} + \Delta^{AD}\Delta^{BC})\,.
\end{aligned}
\tag{68}
$$

3.3.4. Linearized productions

The right-hand side of Eq. (53) may be called production of $M_r^{B_1\cdots B_n}$,

$$
P_r^{B_1\cdots B_n} = -\frac{1}{c\tau}\left\{(M_r^{B_1\cdots B_n} - M_r^{B_1\ldots B_n}{}_{|E})\frac{1}{c^2}V_A U^A + (M_r^{B_1\cdots B_n A} - M_r^{B_1\ldots B_n A}{}_{|E})\frac{1}{c}V_A\right\}. \tag{69}
$$

We ask for the values of the production in case of small deviations from equilibrium. We introduce the small velocity w^A into Eq. (69) and assume that the differences between the moments and their equilibrium values are also small. Keeping only terms which are small in first order we obtain

$$
P_r^{B_1\cdots B_n} = -\frac{1}{c\tau}(M_r^{B_1\cdots B_n} - M_r^{B_1\ldots B_n}{}_{|E})\,. \tag{70}
$$

Note that this and the following formulae for the productions are exact in case that $U^A = V^A$.

Due to the conservation laws of particle number ($P_0 = 0$), energy ($P_1 = 0$) and momentum ($P_1^A = 0$) follows:

$$
\begin{aligned}
&m_0 = M_0 = cn\,,\\
&m_1 = M_1 = e\,,\\
&(m_1 - \tfrac{1}{3} m_1^D{}_D)\frac{1}{c} w_r^A = M_1^A = F^A\,,
\end{aligned}
\tag{71}
$$

where Eq. (49) has been used. Thus, m_0/c and m_1 are the local particle density and energy density, respectively, and w^A is proportional to the momentum density. With $1/3 m_1^D{}_D = -p_{LL} = -nkT$, we may write

$$
w_r^A = \frac{c}{(M_1 + \frac{1}{3}p_{LL})} M_1^A = \frac{c}{(e + nkT)} F^A\,. \tag{72}
$$

Of course, we have $w^A = F^A = 0$, if the theory is based on the Landau–Lifshitz frame.

With Eqs. (68) and (72) we may write the productions for $n=0,1,2$ in terms of the projected moments $M_r^{B_1\cdots B_n}$ as

$$P_r = -\frac{1}{c\tau}\left(M_r - (mc)^r M_0 \frac{I_r}{I_0}\right),$$

$$P_r^A = -\frac{1}{c\tau}\left(M_r^A - (mc)^{r-1}\frac{I_r - r/3(I_{r-2}-I_r)}{I_1 - 1/3(I_{-1}-I_1)} M_1^A\right) \tag{73}$$

$$P_r^{AB} = -\frac{1}{c\tau}\left(M_r^{AB} - \frac{1}{3}(mc)^r M_0 \frac{I_{r-2}-I_r}{I_0}\Delta^{AB}\right),$$

the integrals $I_r = I_r(z)$ are given by Eq. (65b) as functions of inverse temperature $z = mc^2/kT$. Eq. (65a) may now be written as

$$m_r = (mc)^r M_0 \frac{I_r}{I_0},\ r = 2,3,\ldots \quad \text{and} \quad \frac{I_1}{I_0} = \frac{M_1}{mcM_0}, \tag{74}$$

where the last equation defines the temperature.

3.4. The non-relativistic limit

3.4.1. Projected and central moments

In order to interpret the projected moments (48) in the non-relativistic limit we investigate them in the LRF,

$$M_r^{B_1\cdots B_n}{}_{|LR} = c\int (p^0_{|LR})^{r+1-n} R^{B_1}_{|LR}\cdots R^{B_n}_{|LR} f\, dP\,. \tag{75}$$

Because of $R^A_{|LR} = \{0, p^i_{|LR}\}^A$ only the spatial components do not vanish. With $dP = d^3p_{|LR}/p^0_{|LR}$ we obtain for these

$$M_r^{i_1\cdots i_n}{}_{|LR} = c^{1-n}\int (p^0_{|LR})^r \frac{cp^{i_1}_{|LR}}{p^0_{|LR}}\cdots\frac{cp^{i_n}_{|LR}}{p^0_{|LR}} f\, d^3p_{|LR}\,. \tag{76}$$

By Eq. (7) the velocity in LRF is given by

$$C^i = \xi^i_{|LR} = \frac{cp^i_{|LR}}{p^0_{|LR}}\,. \tag{77}$$

$f\,d^3p_{|LR}$ is the number density of particles with momenta in the vicinity of $p^i_{|LR}$ in LRF. We denote the number density of particles with velocities in the vicinity of C^i by $F\,d^3C$, so that

$$F\,d^3C = f\,d^3p_{|LR}\,. \tag{78}$$

Eq. (8) reads in LRF $p^0_{|LR} = mc/\sqrt{1-(C^2/c^2)}$ and we may write instead of Eq. (76)

$$M_r^{i_1\cdots i_n}{}_{|LR} = (mc)^r c^{1-n} \int \frac{1}{\sqrt{1-(C^2/c^2)}^{\,r}} C^{i_1}\cdots C^{i_n} F\, d^3C\,, \tag{79}$$

where the interval of integration is bounded by the speed of light c.

The basic assumption for the non-relativistic limit is that the particles have speeds far below the speed of light, so that

$$M_r^{i_1\cdots i_n}{}_{|LR} \simeq (mc)^r c^{1-n} \int \left(1 + \frac{r}{2}\frac{C^2}{c^2}\right) C^{i_1}\cdots C^{i_n} F\, d^3C + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{80}$$

In this case the phase density F vanishes far below the speed of light, so that the integration may run to infinity now. We define

$$u_k^{i_1\cdots i_n} = m \int C^{2k} C^{i_1}\cdots C^{i_n} F\, d^3C\,, \tag{81}$$

so that

$$M_r^{i_1\cdots i_n}{}_{|LR} \simeq (mc)^{r-1} c^{2-n} \left(u_0^{i_1\cdots i_n} + \frac{1}{c^2}\frac{r}{2} u_1^{i_1\cdots i_n} + \cdots\right). \tag{82}$$

The LRF under consideration is either the Eckart frame with $M_0^A = 0$ or the Landau–Lifshitz frame with $M_1^A = 0$. If only the first-order terms in $1/c$ are considered in Eq. (82), both frames correspond to $u_0^i = m\int C^i F\, d^3C = 0$. Since u_0^i is the non-relativistic momentum density of the gas, we have to interpret C^i as the velocity measured in the rest frame of the gas, where the momentum density vanishes.

Therefore the $u_k^{i_1\cdots i_n}$ are the so-called central moments of the phase density. We give a list of those central moments with physical meaning,

$$\begin{aligned}
&u_0 = m\int F\, d^3C = \varrho, \qquad u_0^i = m\int C^i F\, d^3C = 0,\\
&u_0^{ij} = m\int C^i C^j F\, d^3C = p^{ij}, \qquad u_1 = m\int C^2 F\, d^3C = 2\varrho\varepsilon,\\
&u_1^i = m\int C^2 C^i F\, d^3C = 2q^i.
\end{aligned} \tag{83}$$

Here ϱ is the mass density, p^{ij} denotes the pressure tensor, ε is the specific internal energy and q^i is the heat flux [7].

Obviously, we have $p^i{}_i = 2\varrho\varepsilon$ such that Eq. (83) gives a list of 13 quantities and these are the basic variables of the non-relativistic moment theory, see [18,8].

The natural choice of variables in relativistic moment theory consists of the 14 moments (49) which are related to the non-relativistic central moments (83) by

$$\begin{aligned}
&M_{0\,|\,LR} = cn \simeq \tfrac{c}{m}\varrho, \qquad M^i_{0\,|\,LR} = J^i{}_{|LR} = 0,\\
&M_{1\,|\,LR} = e \simeq \varrho c^2 + \varrho\varepsilon, \qquad M^i_{1\,|\,LR} = F^i{}_{|LR} \simeq \tfrac{1}{c} q^i,\\
&M^i_{1i\,|\,LR} \simeq 2\varrho\varepsilon + \tfrac{1}{c^2}\tfrac{1}{2}u_2, \qquad M_1^{\langle ij\rangle}{}_{|LR} = P^{\langle ij\rangle}{}_{|LR} \simeq p^{\langle ij\rangle}\,.
\end{aligned} \tag{84}$$

Here we have followed Dreyer and Weiss [19] who suggest that the moment $u_2 = m\int C^4 F\, d^3C$ should be considered in non-relativistic theory. The brackets denote the trace-free part of a tensor.

In Eq. (84) we have considered the Eckart frame as the basis for the non-relativistic limit, i.e. we have $M^i_{0\,|\,LR} = 0$ and $M^i_{1\,|\,LR} \neq 0$. If we rely on the Landau–Lifshitz frame and consider terms up to second-order we have $M^i_{0\,|\,LR} \neq 0$ and $M^i_{1\,|\,LL} = mc(M^i_{0\,|\,LL} + (1/mc^2)q^i) = 0$. Thus, we identify the non-relativistic limits of M^A_0 and M^A_1 in the Landau–Lifshitz case by

$$M^i_{0\,|\,LL} = -\frac{1}{mc^2}q^i, \qquad M^i_{1\,|\,LL} = 0\,, \tag{85}$$

while the limits of the other moments are as in Eq. (84).

3.4.2. Moment equations

We ask for the moment equations (53) in a laboratory frame which moves only slowly relative to the gas. The 4-velocity of the gas in the laboratory reads

$$U^A = \frac{1}{\sqrt{1-(v^2/c^2)}}\{c, v^i\}^A\,,$$

where v^i is the 3-velocity of the gas . In the non-relativistic limit we neglect all terms of orders $\mathcal{O}(v^2/c^2)$ so that

$$\begin{aligned}
&U^A = \{c, v^i\}^A, && U_A = \{c, -v_i\}_A\,,\\
&D\Psi = \frac{1}{c}\left(\frac{\partial \Psi}{\partial t} + v^k\frac{\partial \Psi}{\partial x^k}\right) = \frac{1}{c}\frac{D\Psi}{Dt}, && \nabla_C \Psi = \left\{0, \frac{\partial \Psi}{\partial x^i}\right\}_C\,,\\
&\frac{1}{c}DU_A = \left\{0, -\frac{1}{c^2}\frac{\partial v_k}{\partial t}\right\}_A, && \frac{1}{c}\nabla_C U_D = \left\{\begin{matrix} 0 & 0_j \\ 0_i & -\frac{1}{c}\frac{\partial v_k}{\partial x^l}\end{matrix}\right\}_{CD}\,.
\end{aligned} \tag{86}$$

Furthermore, the vector R^A transforms in this limit as

$$R^A = \left\{\frac{v_k}{c}p^k_{|LR}, p^i_{|LR}\right\}^A$$

so that the moments in the laboratory are related to the moments in the rest frame by

$$M_r^{i_1\cdots i_n} = M_r^{i_1\cdots i_n}{}_{|LR}, \qquad M_r^{i_1\cdots i_n 0} = \frac{v_k}{c}M_r^{i_1\cdots i_n k}{}_{|LR}, \qquad M_r^{i_1\cdots i_n 00} = \mathcal{O}\left(\frac{v^2}{c^2}\right) \tag{87}$$

Consideration of the space-time components of Eq. (53), i.e. setting $B_k = i_k$ yields with Eqs. (86), (87) and (82)

$$\frac{Du_0^{i_1\cdots i_n}}{Dt} + \frac{\partial u_0^{i_1\cdots i_n k}}{\partial x^k} + n\frac{\partial v^{(i_1}}{\partial t}u_0^{i_2\cdots i_n)} + \frac{\partial v^k}{\partial x^k}u_0^{i_1\cdots i_n} + n\frac{\partial v_k}{\partial x^l}\delta^{l(i_1}u_0^{i_2\cdots i_n)k}$$

$$+\frac{1}{c^2}\frac{r}{2}\left\{\frac{Du_1^{i_1\cdots i_n}}{Dt}+\frac{\partial u_1^{i_1\cdots i_n k}}{\partial x^k}+n\frac{\partial v^{(i_1}}{\partial t}u_1^{i_2\cdots i_n)}+\frac{\partial v^k}{\partial x^k}u_1^{i_1\cdots i_n}+n\frac{\partial v_k}{\partial x^l}\delta^{l(i_1}u_1^{i_2\cdots i_n)k}\right\}$$

$$+\frac{1}{c^2}\left\{-\frac{\partial v_k}{\partial t}(n-r-1)u_0^{i_1\cdots i_n k}-\frac{\partial v_k}{\partial x^l}(n-r)u_0^{i_1\cdots i_n kl}\right\}$$

$$=-\frac{1}{\tau}(u_0^{i_1\cdots i_n}-u_0^{i_1\cdots i_n}{}_{|E})-\frac{1}{c^2}\frac{r}{2}\frac{1}{\tau}(u_1^{i_1\cdots i_n}-u_1^{i_1\cdots i_n}{}_{|E})\,, \tag{88}$$

where all terms of order $\mathcal{O}(v^2/c^2)$ have been neglected. The equations for the central moments $u_0^{i_1\cdots i_n}$ follow by omission of all terms with $1/c^2$

$$\frac{Du_0^{i_1\cdots i_n}}{Dt}+\frac{\partial u_0^{i_1\cdots i_n k}}{\partial x^k}+n\frac{\partial v^{(i_1}}{\partial t}u_0^{i_2\cdots i_n)}+\frac{\partial v^k}{\partial x^k}u_0^{i_1\cdots i_n}+n\frac{\partial v_k}{\partial x^l}\delta^{l(i_1}u_0^{i_2\cdots i_n)k}$$

$$=-\frac{1}{\tau}(u_0^{i_1\cdots i_n}-u_0^{i_1\cdots i_n}{}_{|E})\,, \tag{89}$$

while the difference of Eq. (88) with $r=0$ and $r=1$ gives the equations for $u_1^{i_1\cdots i_n}$, viz.

$$\frac{Du_1^{i_1\cdots i_n}}{Dt}+\frac{\partial u_1^{i_1\cdots i_n k}}{\partial x^k}+n\frac{\partial v^{(i_1}}{\partial t}u_1^{i_2\cdots i_n)}+\frac{\partial v_k}{\partial x^k}u_1^{i_1\cdots i_n}+n\frac{\partial v_k}{\partial x^l}\delta^{l(i_1}u_1^{i_2\cdots i_n)k}$$

$$=-2\frac{\partial v_k}{\partial t}u_0^{i_1\cdots i_n k}-2\frac{\partial v_k}{\partial x^l}u_0^{i_1\cdots i_n kl}-\frac{1}{\tau}(u_1^{i_1\cdots i_n}-u_1^{i_1\cdots i_n}{}_{|E})\,. \tag{90}$$

Eqs. (89) and (90) are the appropriate moment equations for the central moments (81) of non-relativistic kinetic theory, see [7].

Thus, the projected moment formalism reduces to the formalism of central moments in the non-relativistic limit or, in other words, the projected moment formalism is the relativistic generalization of the formalism of central moments.

4. Relativistic extended thermodynamics with projected moments

4.1. Choice of variables

The objective of relativistic thermodynamics is the determination of the 14 fields of N^A and T^{AB} or, alternatively, of the 14 projected moments number density M_0, energy density M_1, particle flux M_0^A, momentum density M_1^A and pressure tensor M_1^{AB}.

It is customary in relativistic kinetic theory [4,5,7,9] to determine the 14 fields

$$N^A=A_0^A \quad \text{and} \qquad T^{AB}=A_1^{AB}\,,$$

from the 14 moment equations for $A_1^{AB}=T^{AB}$ and A_2^{ABC} - with $A_2^{AB}{}_B=m^2c^2A_0^A=m^2c^2 N^A$ – viz.

$$A_1^{AB}{}_{,B}=0, \tag{91a}$$

$$A_2^{ABC}{}_{,C}=-\frac{1}{c\tau}(A_2^{ABC}-A_2^{ABC}{}_{|E})V_C \quad \text{with } (A_2^A{}_A{}^C-A_2^A{}_A{}^C{}_{|E})V_C=0. \tag{91b}$$

The trace of Eq. (91b) implies the conservation of particle number.

The set (91) of unprojected moment equations is equivalent to the moment equations for the projected moments

$$M_1, M_1^A, M_2, M_2^A, M_2^{AB} \tag{92}$$

and – since the moments M_2, M_2^A, M_2^{AB} have no physical interpretation – this choice of moment equations is somewhat artificial.

To us it seems far more reasonable to determine the moments

$$M_0, M_0^A, M_1, M_1^A, M_1^{AB},$$

from their associated moment equations. These may easily be combined to the 14 equations for the unprojected moments A_0^{AB} and A_1^{ABC} – with $A_0^{AB}\frac{1}{c}U_B=N^A$ and $A_1^{ABC}\frac{1}{c}U_C=T^{AB}$ – which read (46)

$$\begin{aligned} &A_0^{AB}{}_{,B}+A_0^{ABC}\frac{1}{c}U_{C,B} \\ &\quad=-\frac{1}{c\tau}(A_0^{AB}-A_0^{AB}{}_{|E})\frac{1}{c}V_B \quad \text{with } (A_0^{AB}-A_0^{AB}{}_{|E})V_BU_A=0\,, \\ &A_1^{ABC}{}_{,C}+A_1^{ABCD}\frac{1}{c}U_{D,C} \\ &\quad=-\frac{1}{c\tau}(A_1^{ABC}-A_1^{ABC}{}_{|E})\frac{1}{c}V_C \quad \text{with } (A_1^{ABC}-A_1^{ABC}{}_{|E})V_CU_B=0\,. \end{aligned} \tag{93}$$

It should be mentioned that both sets of moment equations (91) and (93), reduce in the non-relativistic limit to the equations (89) and (90) for the central moments u_0, u_1, u_2, u_0^i, u_1^i and u_0^{ij}.

4.2. Moment equations and closure problem

The equations for the projected moments M_0, M_0^A, M_1, M_1^A, M_1^{AB} read, in particular, Balance of particle number

$$DM_0+\nabla_DM_0^D-\frac{1}{c}DU_DM_0^D+\frac{1}{c}\nabla_DU^DM_0=0\,. \tag{94}$$

Balance of energy:

$$DM_1+\nabla_DM_1^D-\frac{2}{c}DU_DM_1^D-\frac{1}{c}\nabla_CU_DM_1^{CD}+\frac{1}{c}\nabla_DU^DM_1=0\,. \tag{95}$$

Balance of particle flux:

$$DM_0^B + \nabla_C M_0^{BC} + \frac{1}{c} DU_D \left\{ \eta^{DB} M_0 + \frac{1}{c} U^B M_0^D \right\}$$
$$+ \frac{1}{c} \nabla_C U_D \left\{ M_0^{BCD} + M_0^B \eta^{CD} + \eta^{DB} M_0^C + \frac{1}{c} U^B M_0^{CD} \right\}$$
$$= -\frac{1}{c\tau} \left(M_0^B - mc \frac{I_0}{I_1 - \frac{1}{3}(I_{-1} - I_1)} M_1^B \right). \tag{96}$$

Balance of momentum:

$$DM_1^B + \nabla_C M_1^{BC} + \frac{1}{c} DU_D \left\{ -M_1^{BD} + \eta^{DB} M_1 + \frac{1}{c} U^B M_1^D \right\}$$
$$+ \frac{1}{c} \nabla_C U_D \left\{ M_1^B \eta^{CD} + \eta^{DB} M_1^C + \frac{1}{c} U^B M_1^{CD} \right\} = 0. \tag{97}$$

Balance of pressure tensor:

$$DM_1^{AB} + \nabla_C M_1^{ABC} + \frac{1}{c} DU_D \left\{ 2\eta^{D(A} M_1^{B)} + 2\frac{1}{c} U^{(A} M_1^{B)D} \right\}$$
$$+ \frac{1}{c} \nabla_C U_D \left\{ M_1^{ABCD} + M_1^{AB} \eta^{CD} + 2\eta^{D(A} M_1^{B)C} + \frac{2}{c} U^{(A} M_1^{B)CD} \right\}$$
$$= -\frac{1}{c\tau} \left(M_1^{AB} - \frac{1}{3} mc M_0 \frac{I_{-1} - I_1}{I_0} \Delta^{AB} \right). \tag{98}$$

It should be kept in mind that the 4-velocity U^A is always meant to be either the Landau–Lifshitz velocity V^A or the Eckart velocity W^A. Thus, it is related to the moments and no additional equation is needed for the determination of U^A. We have either

$$U^A = V^A \quad \text{and} \quad M_1^A = F^A = 0 \quad \text{or}$$
$$U^A = W^A \quad \text{and} \quad M_0^A = J^A = 0\,. \tag{99}$$

The set of 14 equations (94)–(98) does not form a closed set of field equations for the variables

$$M_0,\, M_0^A,\, M_1,\, M_1^A,\, M_1^{AB}\,, \tag{100}$$

because the equations contain the additional moments

$$M_0^{BC},\, M_0^{BCD},\, M_1^{ABC},\, M_1^{ABCD}\,. \tag{101}$$

In order to obtain a closed set of field equations we need constitutive functions which relate the moments (101) to the fields (100). We follow the philosophy of rational extended thermodynamics [8] and search for constitutive equations of the form

$$\begin{aligned} M_0^{BC} &= M_0^{BC}(M_0, M_0^A, M_1, M_1^A, M_1^{AB})\,, \\ M_0^{BCD} &= M_0^{BCD}(M_0, M_0^A, M_1, M_1^A, M_1^{AB})\,, \\ M_1^{ABC} &= M_1^{BCD}(M_0, M_0^A, M_1, M_1^A, M_1^{AB})\,, \\ M_1^{ABCD} &= M_1^{ABCD}(M_0, M_0^A, M_1, M_1^A, M_1^{AB})\,. \end{aligned} \tag{102}$$

Note that gradients or time derivatives are absent from the list of variables. The functions (102) will be determined by means of the entropy maximum principle which is equivalent to the theory of rational extended thermodynamics [15]. Thus, all features of rational extended thermodynamics will be contained in our theory. In particular the resulting field equations will be of symmetric hyperbolic type. This guarantees well-posed Cauchy problems and finite speed of disturbances.

4.3. The strategy of extended thermodynamics

At this place a remark on the strategy of extended thermodynamics is in order: The moment equations (53) form an infinite set of coupled partial differential equations. The moment equation for $M_r^{B_1\cdots B_n}$ contains the moments $M_r^{B_1\cdots B_n C}$ and $M_r^{B_1\cdots B_n CD}$ so that we find an infinite hierarchy of equations with increasing tensorial rank for each value r. In general, the hierarchies for various r are coupled by the right-hand sides of the moment equations, see [13] for the case of radiation. We do not have this coupling in the present case due to the simple interaction term of the BGK model.

Thus the Boltzmann equation is replaced by an infinite set of moment equations for moments with all possible values for the numbers r and n. The assumption of extended thermodynamics is that this infinite set may be truncated at a certain level, for instance at $r=0,\dots,R$ and $n=0,\dots N_r$ with some numbers R and N_r so that one has the equations for the projected moments

$$M_r^{A_1\cdots A_n}, \qquad r=0,\dots,R, \qquad n=0,\dots N_r\,. \tag{103}$$

This set of moment equations requires constitutive equations for the moments

$$M_r^{A_1\cdots A_{N_r}C},\, M_r^{A_1\cdots A_{N_r}CD}, \quad r=0,\dots,R \tag{104}$$

and in extended thermodynamics these constitutive functions are assumed to have the form

$$\begin{aligned} M_r^{A_1\cdots A_{N_r}C} &= M_r^{A_1\cdots A_{N_r}C}(M_s^{B_1\cdots B_m};\ s=0,\dots,R;\ m=0,\dots N_r)\,, \\ M_r^{A_1\cdots A_{N_r}CD} &= M_r^{A_1\cdots A_{N_r}CD}(M_s^{B_1\cdots B_m};\ s=0,\dots,R;\ m=0,\dots N_r)\,. \end{aligned} \tag{105}$$

If a certain process is not satisfactorily described by the resulting set of field equations, one has to increase the numbers of moments R and N_r step by step until the resulting set of field equations describes the process under consideration with sufficient accuracy.

The most simple choice of variables is to choose the numbers r, N_r as $r=0,1$ and $N_0=0, N_1=1$, i.e. to consider the five conserved quantities $cn=M_0$, $e=M_1$ and $F^A=M_1^A$ only. In this case extended thermodynamics – or the entropy maximum principle – gives the equilibrium values for the required constitutive equations $J^A=M_0^A$ and $P^{AB}=M_1^{AB}$, i.e. $J^A=nV^A$ and $P^{AB}=p\Delta_{LL}^{AB}$. The resulting field equations form the relativistic Euler equations, which describe gases in equilibrium.

In this paper the emphasis is laid on the case of 14 moments, $r=0,1$ and $N_0=1$, $N_1=2$, which describes only slowly varying deviations from equilibrium.

4.4. Entropy maximum principle

The definition (48) shows that we will find the required constitutive equations (105) if the phase density f depends on space time only through the variables,

$$f=f(M_s^{B_1\cdots B_m}(x^A),\ s=0,\ldots,R,\ m=0,\ldots N_r;\ p^i)\,. \tag{106}$$

The entropy maximum principle states: the phase density (106) follows by maximization of the entropy density with respect to f under the constraint of prescribed values of

$$M_r^{A_1\cdots A_n}=c\int (p^0_{|LR})^{r+1-n}R^{A_1}\cdots R^{A_n}\,f\,dP,\quad r=0,\ldots,R;\ n=0,\ldots N_r\,. \tag{107}$$

For reasons of simplicity we maximize the entropy in LRF where it reads

$$\eta=\frac{1}{c^2}S^AU_A=-k\int p^0_{|LR}f\,\ln\frac{f}{y}\,dP\,. \tag{108}$$

We take care of the constraints by Lagrange multipliers and maximize

$$-k\int p^0_{|LR}f\,\ln\frac{f}{y}\,dP$$
$$-\sum_{r=0}^{R}\sum_{n=0}^{N_r}\Lambda^r_{A_1\cdots A_n}\left(c\int(p^0_{|LR})^{r+1-n}R^{A_1}\cdots R^{A_n}\,f\,dP-M_r^{A_1\cdots A_n}\right)$$

without constraints. The result reads

$$f=y\exp\Sigma\quad\text{with}\quad \Sigma=-1-\sum_{r=0}^{R}\sum_{n=0}^{N_r}\Lambda^r_{A_1\cdots A_n}\frac{c}{k}(p^0_{|LR})^{r-n}R^{A_1}\cdots R^{A_n}\,, \tag{109}$$

where the Λ's are functions of $M_s^{B_1\cdots B_m}$ and follow from the constraints.

Unfortunately, it is impossible to perform the required integrals over the phase density (109). For this reason, we shall expand it around equilibrium. The equilibrium phase density (26) may be written as

$$f_{|E} = y \exp \Sigma_{|E} \quad \text{with} \quad \Sigma_{|E} = -1 - \frac{1}{k}[\lambda_n + \lambda_e p^0_{|LR} + \lambda_A R^A] \tag{110}$$

and comparison with Eq. (109) shows that we have for the Lagrange multipliers in equilibrium

$$\Lambda^0_{|E} = \frac{1}{c}\lambda_n, \quad \Lambda^1_{|E} = \frac{1}{c}\lambda_e, \quad \Lambda^1_{A|E} = \frac{1}{c}\lambda_A, \quad \Lambda^r_{A_1\cdots A_n} = 0 \text{ (for all other } r, n) . \tag{111}$$

We write

$$\Lambda^r_{A_1\cdots A_n} = \Lambda^r_{A_1\cdots A_n\,|E} + \frac{k}{c}\lambda^r_{A_1\cdots A_n} \tag{112}$$

and assume that the non-equilibrium parts of the Lagrange multipliers $\lambda^r_{A_1\cdots A_n}$ are small such that

$$f = f_{|E}\left(1 - \sum_{r=0}^{R}\sum_{n=0}^{N_r} \lambda^r_{A_1\cdots A_n}\frac{c}{k}(p^0_{|LR})^{r-n} R^{A_1}\cdots R^{A_n}\right) . \tag{113}$$

In the next step, the Lagrange multipliers have to be determined from the constraints (63). This will be done in the next sections for the 14 moment case.

4.5. Phase density in the 14 moment case

In the case of 14 moments, $r = 0, 1$ and $N_0 = 1, N_1 = 2$, the phase density (113) reads

$$f = f_{|E}\left(1 - \lambda^0 - \lambda^0_A \frac{R^A}{p^0_{|LR}} - \lambda^1 p^0_{|LR} - \lambda^1_A R^A - \lambda^1_{AB}\frac{R^A R^B}{p^0_{|LR}}\right) \tag{114}$$

or, with $\alpha_A = (\lambda^0/c)U_A + \lambda^0_A$ and $\beta_{AB} = (\lambda^1/c^2)U_A U_B + (1/c)\lambda^1_{(A}U_{B)} + \lambda^1_{AB}$,

$$f = f_{|E}\left(1 - \alpha_A \frac{p^A}{p^0_{|LR}} - \beta_{AB}\frac{p^A p^B}{p^0_{|LR}}\right). \tag{115}$$

This non-equilibrium phase density seems to be new in the literature. In [1–9] the authors use a non-equilibrium phase density of the form

$$f = f_{|E}(1 - \hat{\alpha}_A p^A - \hat{\beta}_{AB} p^A p^B) , \tag{116}$$

a form which follows from the entropy maximum principle if the theory is based on the unprojected moments $M_r^{A_1\cdots A_n}$ with $r = 1, 2$ and $N_1 = 1, N_2 = 2$.

Thus, we do not only propose an alternative set of moment equations in this paper but we propose also an alternative phase density for the closure of the equations.

4.6. Lagrange multipliers in the 14 moment case

For the explicit calculations it is convenient to write instead of Eq. (114)

$$f = f_{|E}\left(1 - \lambda^{-1}\frac{1}{p^0_{|LR}} - \lambda^0 - \hat{\lambda}^1 p^0_{|LR} - \lambda^0_A \frac{R^A}{p^0_{|LR}} - \lambda^1_A R^A - \lambda^1_{\langle AB\rangle}\frac{R^{\langle A}R^{B\rangle}}{p^0_{|LR}}\right), \tag{117}$$

where $\lambda^{-1} = -\frac{1}{3}\lambda^{1\,A}_A$, $\hat{\lambda}^1 = \lambda^1 + \frac{1}{3}\lambda^{1\,A}_A$. The brackets denote the trace-free part of a tensor, for instance, $R^{\langle A}R^{B\rangle} = R^A R^B + \frac{1}{3}R^D R_D \Delta^{AB}$.

With the phase density (117) we calculate the moments M_0, M_0^A, M_1, M_1^A, M_1^{AB} and obtain the Lagrange multipliers as, see Appendix B for details of the calculation,

$$\frac{\lambda^{-1}}{mc} = \frac{I_0 I_2 - I_1^2}{I_{-2}I_0I_2 + 2I_{-1}I_1I_0 - I_{-2}I_1^2 - I_{-1}^2 I_2 - I_0^3}\frac{3(p - p_{|E})}{mc(M_0/I_0)},$$

$$\lambda^0 = \frac{I_0 I_1 - I_2 I_{-1}}{I_{-2}I_0I_2 + 2I_{-1}I_1I_0 - I_{-2}I_1^2 - I_{-1}^2 I_2 - I_0^3}\frac{3(p - p_{|E})}{mc(M_0/I_0)}, \tag{118}$$

$$\hat{\lambda}^1 mc = \frac{I_{-1} I_1 - I_0^2}{I_{-2}I_0I_2 + 2I_{-1}I_1I_0 - I_{-2}I_1^2 - I_{-1}^2 I_2 - I_0^3}\frac{3(p - p_{|E})}{mc(M_0/I_0)},$$

$$\lambda^{0A} = \frac{-3(I_0 - I_2)}{(I_0 - I_2)(I_{-2} - I_0) - (I_{-1} - I_1)^2}\frac{I_0}{M_0}\left(M_0^A - \frac{I_0}{mc(I_1 - \frac{1}{3}(I_{-1} - I_1))}M_1^A\right), \tag{119}$$

$$mc\lambda^{1A} = \frac{3(I_{-1} - I_1)}{(I_0 - I_2)(I_{-2} - I_0) - (I_{-1} - I_1)^2}\frac{I_0}{M_0}\left(M_0^A - \frac{I_0}{mc(I_1 - \frac{1}{3}(I_{-1} - I_1))}M_1^A\right),$$

$$m^2c^2\lambda^{1\langle AB\rangle} = -\frac{15}{2}\frac{I_0/M_0}{I_{-2} - 2I_0 + I_2}M_1^{\langle AB\rangle}. \tag{120}$$

Here p and $p_{|E}$ stand for the pressure $p = -\frac{1}{3}M_1^A{}_A = \frac{1}{3}(M_1 - m^2c^2M_{-1})$ and the equilibrium pressure $p_{|E} = -\frac{1}{3}m_1^A{}_A = \frac{1}{3}(m_1 - m^2c^2m_{-1})$, respectively. The combination $p - p_{|E} = (m^2c^2/3)(m_{-1} - M_{-1})$ is called dynamical pressure.

Eqs. (118)–(120) are valid only in first order in deviations from equilibrium.

4.7. Constitutive equations

Now, we are able to calculate the constitutive equations (102) for the moments M_0^{AB}, M_0^{ABC}, M_1^{ABC} and M_1^{ABCD}. For reasons of simplicity we write the moments as sums of their traces and trace-free parts,

$$M_r^{AB} = M_r^{\langle AB\rangle} + \tfrac{1}{3}\Delta^{AB} M_r{}^D{}_D ,$$

$$M_r^{ABC} = M_r^{\langle ABC\rangle} + \tfrac{1}{5}(\Delta^{AB} M_r^{CD}{}_D + \Delta^{AC} M_r^{BD}{}_D + \Delta^{BC} M_r^{AD}{}_D) ,$$

$$\begin{aligned} M_r^{ABCD} = {} & M_r^{\langle ABCD\rangle} \\ & + \tfrac{1}{7}(M_r^{\langle AB\rangle G}{}_G \Delta^{CD} + M_r^{\langle AC\rangle G}{}_G \Delta^{BD} + M_r^{\langle AD\rangle G}{}_G \Delta^{BC} \\ & + M_r^{\langle BC\rangle G}{}_G \Delta^{AD} + M_r^{\langle BD\rangle G}{}_G \Delta^{AC} + M_r^{\langle CD\rangle G}{}_G \Delta^{AB}) \\ & + \tfrac{1}{15} M_r{}^F{}_F{}^G{}_G (\Delta^{AB}\Delta^{CD} + \Delta^{AC}\Delta^{BD} + \Delta^{AD}\Delta^{BC}) , \end{aligned} \tag{121}$$

where the brackets denote symmetric trace-free tensors, see [13] for details. The traces are related to the moments of lower rank by Eq. (52)

$$\begin{aligned} M_r{}^D{}_D &= m^2c^2 M_{r-2} - M_r , \\ M_r^{AD}{}_D &= m^2c^2 M_{r-2}^A - M_r^A , \\ M_r^{\langle AB\rangle G}{}_G &= m^2c^2 M_{r-2}^{\langle AB\rangle} - M_r^{\langle AB\rangle} , \\ M_r{}^F{}_F{}^G{}_G &= m^4c^4 M_{r-4} - 2m^2c^2 M_{r-2} + M_r . \end{aligned} \tag{122}$$

Therefore, the required constitutive relations (102) follow from the knowledge of the moments:

$$M_{-2}, M_{-2}^A, M_{-1}^A, M_{-1}^{\langle AB\rangle}, M_0^{\langle AB\rangle}, M_0^{\langle ABC\rangle}, M_1^{\langle ABC\rangle}, M_1^{\langle ABCD\rangle} \tag{123}$$

as function of the moments (100) or as functions of

$$M_{-1}, M_0, M_1, M_0^A, M_1^A, M_1^{\langle AB\rangle} , \tag{124}$$

where we have replaced M_1^{AB} by its trace-free part $M_1^{\langle AB\rangle}$ and its trace $M_1{}^D{}_D = m^2c^2 M_{-1} - M_1$.

Again, we consider only the first-order deviations from equilibrium and obtain from the definition of projected moments (48) and the distribution functions (117)–(120) the moments with index r as

$$\begin{aligned} M_r = {} & (mc)^r M_0 \frac{I_r}{I_0} - 3(p - p_{|E})(mc)^{r-1} \\ & \times \frac{(I_0 I_2 - I_1^2) I_{r-1} + (I_0 I_1 - I_{-1} I_2) I_r + (I_{-1} I_1 - I_0^2) I_{r+1}}{I_{-2} I_0 I_2 + 2 I_{-1} I_1 I_0 - I_{-2} I_1^2 - I_{-1}^2 I_2 - I_0^3} , \end{aligned}$$

$$\frac{M_r^A}{(mc)^r} = \left(\frac{(I_{r-2} - I_r)(I_0 - I_2) - (I_{r-1} - I_{r+1})(I_{-1} - I_1)}{(I_0 - I_2)(I_{-2} - I_0) - (I_{-1} - I_1)^2} \right) M_0^A$$

$$+ \left(\frac{I_r - \frac{r}{3}(I_{r-2} - I_r)}{I_1 - \frac{1}{3}(I_{-1} - I_1)} - \frac{I_0}{I_1 - \frac{1}{3}(I_{-1} - I_1)} \right.$$

$$\left. \times \frac{(I_{r-2} - I_r)(I_0 - I_2) - (I_{r-1} - I_{r+1})(I_{-1} - I_1)}{(I_0 - I_2)(I_{-2} - I_0) - (I_{-1} - I_1)^2} \right) \frac{M_1^A}{mc} , \qquad (125)$$

$$M_r^{\langle AB \rangle} = (mc)^{r-1} \frac{I_{r-3} - 2I_{r-1} + I_{r+1}}{I_{-2} - 2I_0 + I_2} M_1^{\langle AB \rangle} ,$$

$$M_r^{\langle ABC \rangle} = 0 ,$$

$$M_r^{\langle ABCD \rangle} = 0 .$$

Eq. (121) together with Eqs. (122) and (125) give the desired constitutive equations for the moments (102). If the constitutive equations are inserted into the moment equations (94)–(98) one obtains a system of field equations for the moments (100). Again we point out that either M_0^A or M_1^A is equal to zero by Eq. (99).

Since the system was closed by means of rational extended thermodynamics, the resulting field equations form a set of symmetric hyperbolic equations. In this paper we do not further investigate the properties of the system.

5. Local thermal equilibrium

5.1. Conservation laws

In this section we restrict the attention to the projected moments which are defined in the Landau–Lifshitz frame, i.e. we set $U^A = V^A$. The Landau–Lifshitz frame has the advantage that the equilibrium phase density is isotropic with respect to the vector R^A.

The five conservation laws for number density M_0, momentum density M_1^A and energy density M_1 read in this frame

$$DM_0 + \nabla_D M_0^D - \frac{1}{c} M_0^D DV_D + \frac{1}{c} M_0 \nabla_D V^D = 0 ,$$

$$\frac{1}{c}(\eta^{BD} M_1 - M_1^{BD}) DV_D + \nabla_D M_1^{BD} + \frac{1}{c^2} V^B M_1^{CD} \nabla_C V_D = 0 , \qquad (126)$$

$$DM_1 - \frac{1}{c} M_1^{CD} \nabla_C V_D + \frac{1}{c} M_1 \nabla_D V^D = 0 ,$$

since $M_1^A = 0$ in this case. Therefore, the momentum balance (126) may be considered as an equation for the Landau–Lifshitz velocity V^A instead as an equation for the momentum density.

Thus, we have five equations for the five unknowns M_0, M_1 and V^A. The set (126) of equations is not closed unless we provide constitutive equations for M_0^A and M_1^{AB}.

If we follow the philosophy of extended thermodynamics these constitutive equations have the form

$$M_0^A = M_0^A(M_0, M_1), \qquad M_1^{AB} = M_1^{AB}(M_0, M_1)\,.$$

Application of the entropy maximum principle leads the equilibrium phase density in this case and we obtain the result that M_0^A and M_1^{AB} are given by their equilibrium values, see Eq. (63),

$$M_0^A = 0, \qquad M_1^{AB} = m_1^{AB} = \tfrac{1}{3} m_1{}^D{}_D \Delta^{AB} = -p_{|E}\Delta^{AB}\,. \tag{127}$$

Eq. (126) with Eq. (127) may be referred to as the relativistic Euler equations. They read

$$\begin{aligned} &DM_0 + \frac{1}{c} M_0 \nabla_D V^D = 0\,,\\ &(M_1 + p_{|E})\frac{1}{c} DV^B - \nabla^B p_{|E} = 0\,,\\ &DM_1 + (M_1 + p_{|E})\frac{1}{c}\nabla_D V^D = 0\,, \end{aligned} \tag{128}$$

where we have used that

$$\nabla_C(\Delta^{BC}) + \Delta^{CD}\frac{1}{c^2} V^B \nabla_C V_D = 0\,. \tag{129}$$

Since, by Eqs. (65a), (65b) and (120), $M_1 = mcM_0 I_1/I_0$ holds, where the integrals depend on the single variable $z = mc^2/kT$, we may write the energy balance (128) as

$$-mcM_0 \frac{d(I_1/I_0)}{dz}\frac{z}{T} DT + p_{|E}\frac{1}{c}\nabla_D V^D = 0\,. \tag{130}$$

5.2. Maxwell iteration (M_0^A, M_1^{AB})

The Euler equations are appropriate for processes where the local phase density equals the local equilibrium phase density. The first deviations from this case are easily obtained by a so-called Maxwell iteration [7]. The procedure works as follows: We consider the moment equations (53) and insert the equilibrium values of the moments (68) on the left-hand sides of these equations. Solving for the right hand sides gives the first iterates. If one is interested in iterates of higher order one may insert the first iterates in the left-hand sides and solve – again – for the moments on the right-hand side.

Here, we are only interested in the first iterates for the moments M_0^A and M_1^{AB}. Therefore, we need only consider the Eqs. (96) and (98). Since, we are in the Landau–Lifshitz frame we have $w^A = 0$ and obtain

$$M_0 \frac{1}{c} DV^B + \frac{1}{3}\nabla^B m_0{}^A{}_A = -\frac{1}{c\tau} M_0^B\,, \tag{131}$$

$$\Delta^{AB} D p_{|E} - \left(\frac{1}{15} m_1^F{}_F{}^G{}_G - p_{|E}\right)\left(\frac{1}{c}\nabla^B V^A + \frac{1}{c}\nabla^A V^B + \Delta^{AB}\frac{1}{c}\nabla_D V^D\right)$$
$$= \frac{1}{c\tau}(M_1^{AB} + p_{|E}\Delta^{AB}), \tag{132}$$

where we have used Eq. (129) and the identity

$$D\Delta^{AB} + \frac{1}{c^2}(V^A D V^B + V^B D V^A) = 0 . \tag{133}$$

The Euler equations help to eliminate the time derivatives and after some algebra the following constitutive laws for M_0^B, $M_1^{\langle AB\rangle}$ and $p = -\frac{1}{3}M_1^D{}_D$ are obtained:

$$\alpha\nabla^B M_0 - \beta\frac{5}{2}\frac{1}{z^2}\frac{M_0}{T}\nabla^B T = \frac{1}{c\tau}M_0^B , \tag{134a}$$

$$\gamma 2 p_{|E}\frac{1}{c}\nabla^{\langle A}V^{B\rangle} = \frac{1}{c\tau}M_1^{\langle AB\rangle} , \tag{134b}$$

$$-\delta\frac{p_{|E}}{3}\frac{1}{c}\nabla_D V^D = \frac{1}{c\tau}(p - p_{|E}) . \tag{134c}$$

Here the coefficients α to δ depend on $z = mc^2/kT$ by

$$\begin{aligned}
\alpha &= \frac{1}{3}\frac{I_0 - I_{-2}}{I_0} - \frac{I_1 - I_{-1}}{4I_1 - I_{-1}} ,\\
\beta &= \frac{2}{5}z^2\left[\frac{I_1 - I_{-1}}{4I_1 - I_{-1}} - \frac{z}{3}\frac{1}{I_0}\left(\frac{2}{z}(I_{-2} - I_{-4}) - I_{-3} + \frac{I_{-2}}{I_0}I_{-1}\right)\right] ,\\
\gamma &= \frac{4I_1 - 3I_{-1} - I_{-3}}{5(I_1 - I_{-1})} ,\\
\delta &= \frac{I_1 - I_{-3}}{I_1 - I_{-1}} + \frac{3}{z^2}\frac{I_0}{\frac{1}{z}(I_{-1} - I_1) - I_0 + \frac{I_1}{I_0}I_{-1}} .
\end{aligned} \tag{135}$$

where we have used Eq. (67).

Since by Eq. (85) M_0^B is the relativistic equivalent to the heat flux, we may speak of Eq. (134a) as the relativistic law of Fourier. The term with the gradient of M_0 is a purely relativistic one and vanishes in the non-relativistic limit $z \to \infty$, because of $\lim_{z\to\infty}\alpha = 0$. On the other hand we have $\lim_{z\to\infty}\beta = 1$ and (134a) reduces to the well-known Fourier law of non-relativistic theory, $q_i = -\tau\frac{5}{2}\varrho k^2/m^2 T\partial T/\partial x_i$ in this limit. Moreover, we find $\alpha = \frac{1}{12}$ and $\beta = 0$ in the ultra-relativistic limit $z \to 0$.

Eq. (134b) gives the relativistic counterpart to the Navier–Stokes law. In the non-relativistic limit we have $\lim_{z\to\infty}\gamma = 1$ and obtain $p^{\langle ij\rangle} = -\tau 2p(\partial v^{\langle i}/\partial x_{j\rangle})$; the ultra-relativistic case gives $\lim_{z\to 0}\gamma = \frac{4}{5}$.

Eq. (134c) for the dynamic pressure $p - p_{|E}$ has no non-relativistic counterpart because of $\lim_{z\to\infty}\delta = 0$, so that $p = p_{|E} = nkT$ in the non-relativistic limit. In the

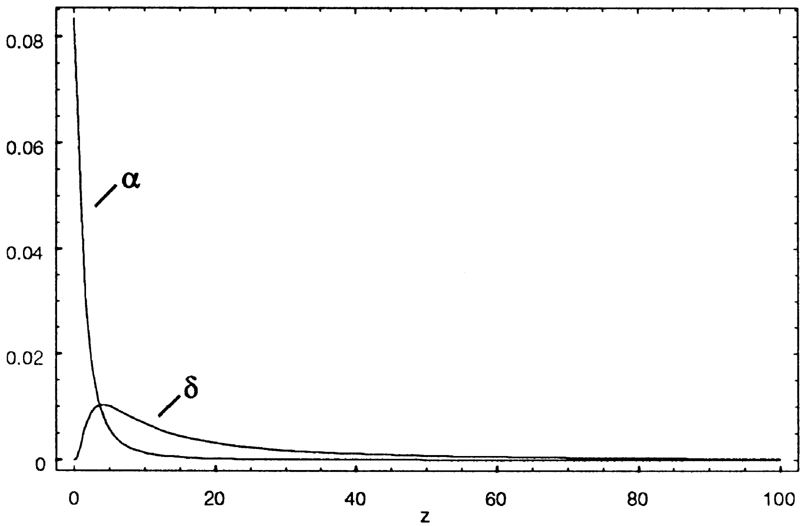

Fig. 1. The coefficients α and δ as functions of inverse temperature $z = mc^2/kT$.

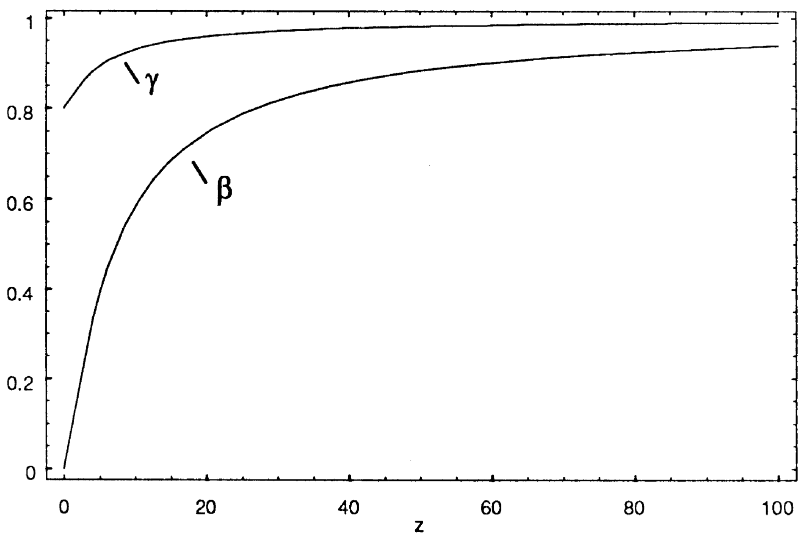

Fig. 2. The coefficients β and γ as functions of inverse temperature $z = mc^2/kT$.

ultra-relativistic case $z \to 0$ we have $\lim_{z\to 0} \delta = 0$ and recover the well-known relation $p = p_{|E} = \frac{1}{3}M_1$ between pressure and energy density.

In Figs. 1 and 2 we show the detailed dependence of the coefficients α to δ on the dimensionless inverse temperature z; the integrals were solved numerically.

5.3. Chapman–Enskog method

The relativistic BGK equation (32) may be rewritten as

$$f = f_{|E} - \frac{c\tau}{p^0_{|LL}}\, p^A f_{,A} \tag{136}$$

and the basic assumption of the Chapman–Enskog method is that f on the right-hand side may be replaced by $f_{|E}$, such that

$$f = f_{|E} - \frac{c\tau}{p^0_{|LL}}\, p^A (f_{|E})_{,A}\,. \tag{137}$$

This phase density may be used to calculate the constitutive functions for $M_0^B, M_1^{\langle AB\rangle}$ and $p = -\frac{1}{3}M_1^D{}_D$. Here we will not perform the calculations in detail, but we will give arguments why this method will give the same results that were obtained from the Maxwell iteration.

We calculate the moment

$$M_r^{A_1\cdots A_n} = c\int (p^0_{|LL})^{r+1-n} R^{A_1}\cdots R^{A_n} f\, dP \tag{138}$$

from the Chapman–Enskog phase density (137). Insertion of Eq. (137) into Eq. (138) yields

$$\begin{aligned} M_r^{A_1\cdots A_n} &= c\int (p^0_{|LL})^{r+1-n} R^{A_1}\cdots R^{A_n}\left(f_{|E} - \frac{c\tau}{p^0_{|LL}}\, p^A (f_{|E})_{,A}\right) dP\,, \\ M_r^{A_1\cdots A_n} &= M_r^{A_1\cdots A_n}{}_{|E} - c^2\tau \int (p^0_{|LL})^{r-n} R^{A_1}\cdots R^{A_n} p^A (f_{|E})_{,A}\, dP\,, \end{aligned} \tag{139}$$

since τ is a constant.

The moment equation for $M_r^{A_1\cdots A_n}$ follows from multiplication of the relativistic Boltzmann equation (32) by $c(p^0_{|LL})^{r-n} R^{A_1}\cdots R^{A_n}\, dP$ and subsequent integration and may therefore be written as

$$c\int (p^0_{|LL})^{r-n} R^{A_1}\cdots R^{A_n} p^A f_{,A}\, dP = -\frac{1}{c\tau}\left(M_r^{A_1\cdots A_n} - M_r^{A_1\cdots A_n}{}_{|E}\right). \tag{140}$$

We perform the Maxwell iteration by replacing of f by $f_{|E}$ on the left-hand side of Eq. (140). A little reorganization gives Eq. (139) – the same result as the Chapman–Enskog method.

Note that both methods give the same result only in the case that τ does not depend on p^A. The case of velocity-dependent relaxation times is discussed for the non-relativistic case in [20]. In this paper it is shown that the moment method with a large number of moments gives always the same results as the Chapman–Enskog method. This is the reason why we consider the Chapman–Enskog method as a benchmark for the moment method.

5.4. Maxwell iteration (M_2^A, M_2^{AB})

5.4.1. Objective

In this subsection we calculate the laws of Fourier and Navier–Stokes with a Maxwell iteration in the moment equations for the moments M_2^A and M_2^{AB}. Again we insert the equilibrium moments on the left-hand sides and eliminate the time derivatives by means of the Euler equations. The result reads

$$\alpha_2 \nabla^B M_0 - \beta_2 \frac{5}{2}\frac{1}{z^2}\frac{M_0}{T}\nabla^B T = \frac{1}{c\tau}\frac{M_2^B}{m^2c^2},$$

$$\gamma_2 2p_{|E}\frac{1}{c}\nabla^{\langle A}V^{B\rangle} = \frac{1}{c\tau}\frac{M_2^{\langle AB\rangle}}{mc}, \tag{141}$$

$$-\delta_2\frac{p_{|E}}{3}\frac{1}{c}\nabla_D V^D = -\frac{1}{c\tau}\frac{M_2^D{}_D - m_2^D{}_D}{3mc}$$

with the temperature-dependent coefficients

$$\alpha_2 = \frac{1}{3}\frac{I_2 - I_0}{I_0} - \frac{1}{3}\frac{I_1 - I_{-1}}{4I_1 - I_{-1}}\frac{5I_2 - 2I_0}{I_0},$$

$$\beta_2 = \frac{2}{15}z^2\left[\frac{I_1 - I_{-1}}{4I_1 - I_{-1}}\frac{5I_2 - 2I_0}{I_0} - \frac{z}{I_0}\left(-I_{-1} - \frac{2}{z}I_0 + I_1 + \frac{2}{z}I_2 + \frac{I_0 - I_2}{I_0}I_{-1}\right)\right], \tag{142}$$

$$\gamma_2 = \frac{I_2 - I_0}{I_1 - I_{-1}},$$

$$\delta_2 = 2\frac{I_2 - I_0}{I_1 - I_{-1}} + \frac{-I_{-1} - \frac{2}{z}I_0 + I_1 + \frac{2}{z}I_2 + (I_0 - I_2)/I_0 I_{-1}}{\frac{1}{z}(I_{-1} - I_1) - I_0 + I_1/I_0 I_{-1}}.$$

In order to obtain constitutive equations for the moments M_0^A and M_1^{AB} we need relations between these and the moments M_2^A and M_2^{AB} which appear on the right-hand sides of Eq. (141). These relations must be of the form

$$\frac{M_2^B}{m^2c^2} = \sigma M_0^B, \qquad \frac{M_2^{\langle AB\rangle}}{mc} = \mu M_2^{\langle AB\rangle}, \qquad \frac{M_2^D{}_D - m_2^D{}_D}{3mc} = -\nu(p - p_{|E}) \tag{143}$$

and we will use the phase densities (115) and (116) for the determination of the dimensionless coefficients σ, μ and ν and compare the results. With Eq. (143) we obtain from Eq. (141)

$$\frac{\alpha_2}{\sigma}\nabla^B M_0 - \frac{\beta_2}{\sigma}\frac{5}{2}\frac{1}{z^2}\frac{M_0}{T}\nabla^B T = \frac{1}{c\tau}M_0^B\,,$$

$$\frac{\gamma_2}{\mu}2p_{|E}\frac{1}{c}\nabla^{\langle A}V^{B\rangle} = \frac{1}{c\tau}M_1^{\langle AB\rangle}\,, \tag{144}$$

$$-\frac{\delta_2}{\nu}\frac{p_{|E}}{3}\frac{1}{c}\nabla_D V^D = \frac{1}{c\tau}(p - p_{|E})\,.$$

and comparison with Eq. (134a)–(134c) shows that we have to compare the pairs

$$\left(\alpha, \frac{\alpha_2}{\sigma}\right), \quad \left(\beta, \frac{\beta_2}{\sigma}\right), \quad \left(\gamma, \frac{\gamma_2}{\mu}\right), \quad \left(\delta, \frac{\delta_2}{\nu}\right).$$

5.4.2. Closure by entropy maximization

We start with the phase density (115) which was derived from the entropy maximum principle in this paper. The desired relations (143) were already calculated in Section 4.7 and we may identify the coefficients in this case as

$$\sigma_{EMP} = \frac{(I_0 - I_2)^2 - (I_{-1} - I_1)(I_1 - I_3)}{(I_0 - I_2)(I_{-2} - I_0) - (I_{-1} - I_1)^2}\,,$$

$$\mu_{EMP} = \frac{I_{-1} - 2I_1 + I_3}{I_{-2} - 2I_0 + I_2}\,, \tag{145}$$

$$\nu_{EMP} = \frac{(I_0 I_2 - I_1^2)I_1 + (I_0 I_1 - I_{-1}I_2)I_2 + (I_{-1}I_1 - I_0^2)I_3}{I_{-2}I_0 I_2 + 2I_{-1}I_1 I_0 - I_{-2}I_1^2 - I_{-1}^2 I_2 - I_0^3}\,.$$

The comparison with the former result must be performed numerically and it yields the interesting result

$$\frac{\alpha_2}{\sigma_{EMP}} = \alpha, \quad \frac{\beta_2}{\sigma_{EMP}} = \beta, \quad \frac{\gamma_2}{\mu_{EMP}} = \gamma, \quad \frac{\delta_2}{\nu_{EMP}} = \delta\,, \tag{146}$$

i.e. use of the moment equations for M_2^A and M_2^{AB} and the phase density (115) yields the same results as the Chapman–Enskog method or the use of the moment equations for M_0^A and M_1^{AB}.

In order to find an explanation for this we rewrite the Chapman–Enskog phase density (137) with Eq. (27) as

$$f = f_{|E}\left(1 - (\ln \mathscr{A})_{,A}\frac{p^A}{p^0_{|LL}} + c\tau(\beta V_B)_{,A}\frac{p^A p^B}{p^0_{|LL}}\right). \tag{147}$$

This function has the same dependence on p^A and $p^0_{|LL}$ as the phase density (115) and we suppose that this is the reason for this coincidence.

5.4.3. Closure with Chernikov's phase density

Chernikov's phase density (116) yields a different set of coefficients for Eq. (143). From the calculations of Appendix C, especially from Eqs. (C.7)–(C.9) we find

$$\sigma_{Ch} = \frac{(I_1 - I_3)^2 - (I_0 - I_2)(I_2 - I_4)}{(I_{-1} - I_1)(I_1 - I_3) - (I_0 - I_2)^2},$$

$$\mu_{Ch} = \frac{I_0 - 2I_2 + I_4}{I_{-1} - 2I_1 + I_3}, \tag{148}$$

$$\nu_{Ch} = \frac{(I_1 I_3 - I_2^2)I_2 + (I_1 I_2 - I_0 I_3)I_3 + (I_0 I_2 - I_1^2)I_4}{I_{-1}I_1 I_3 + 2I_0 I_1 I_2 - I_{-1}I_2^2 - I_0^2 I_3 - I_1^3}.$$

We have to compare these with the coefficients (145), and the best measures for the comparison are the relative errors,

$$E_\sigma = 1 - \frac{\sigma_{Ch}}{\sigma_{EMP}}, \qquad E_\mu = 1 - \frac{\mu_{Ch}}{\mu_{EMP}}, \qquad E_\nu = 1 - \frac{\nu_{Ch}}{\nu_{EMP}}. \tag{149}$$

These functions are plotted in Fig. 3 as functions of inverse temperature. It is clearly seen that the errors vanish in the non-relativistic limit $z \to \infty$, while they are biggest in the ultra-relativistic limit $z \to 0$. We have to conclude, that Chernikov's phase density is not suitable in the case under consideration.

5.5. Application of Marle's BGK model

We consider the BGK model of Marle [4], see Eq. (35). We are interested in the relativistic Navier–Stokes law only and consider the moment equations for $M_1^{\langle AB\rangle}$ and $M_2^{\langle AB\rangle}$. Again we perform the Maxwell iteration and obtain

$$\gamma 2p_{|E}\frac{1}{c}\nabla^{\langle A}V^{B\rangle} = \frac{m}{\tau}M_0^{\langle AB\rangle}, \tag{150a}$$

$$\gamma_2 2p_{|E}\frac{1}{c}\nabla^{\langle A}V^{B\rangle} = \frac{1}{c\tau}M_1^{\langle AB\rangle}; \tag{150b}$$

note that the moment equation for $M_r^{\langle AB\rangle}$ has the moment $M_{r-1}^{\langle AB\rangle}$ on the right-hand side now. It follows that it is the moment equation for $M_2^{\langle AB\rangle}$ (150b) (with $M_1^{\langle AB\rangle}$ on the

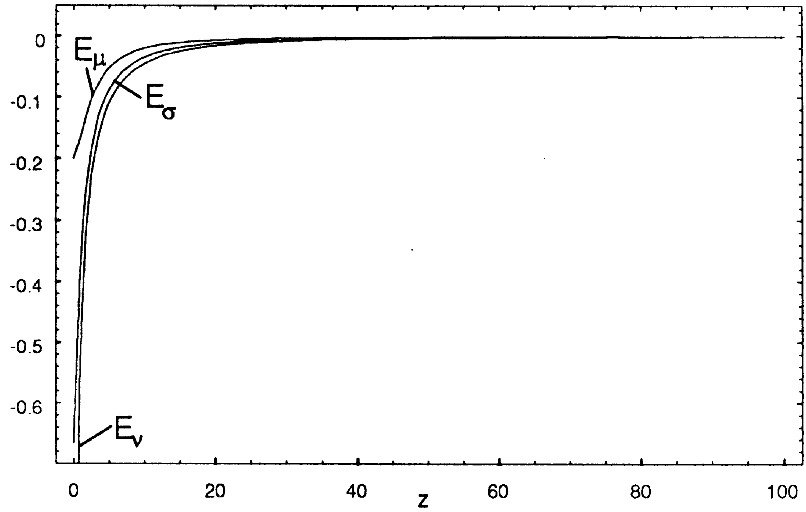

Fig. 3. Relative errors E_σ, E_μ and E_v as functions of inverse temperature z. E_v goes down to -3 in the limit $z \longrightarrow 0$.

right-hand side) which gives the same result as the Chapman–Enskog method. In order to compare the two phase densities, we have to find a relation

$$M_0^{\langle AB\rangle} = \chi \frac{M_1^{\langle AB\rangle}}{mc} \tag{151}$$

for the use in Eq. (150a). From Eqs. (125) and (C.9) follows:

$$\chi_{EMP} = \frac{I_{-3} - 2I_{-1} + I_1}{I_{-2} - 2I_0 + I_2}, \qquad \chi_{Ch} = \frac{I_{-2} - 2I_0 + I_2}{I_{-1} - 2I_1 + I_3} \tag{152}$$

and we have to compare γ/χ_{EMP} and γ/χ_{Ch} with the Chapman–Enskog result γ_2. A numerical analysis shows that

$$\frac{\gamma}{\chi_{Ch}} = \gamma_2$$

and it follows that Chernikov's phase density is appropriate in Marle's BGK model. This is due to the fact that the application of the Chapman–Enskog method to Marle's BGK model would give a phase density which depends on p^A and $p^0_{|LL}$ in the same way as Chernikov's phase density.

Fig. 4 shows the relative error

$$E_\chi = 1 - \frac{\chi_{EMP}}{\chi_{Ch}}$$

and it is clearly seen that the phase density which was introduced in this paper is not suitable here.

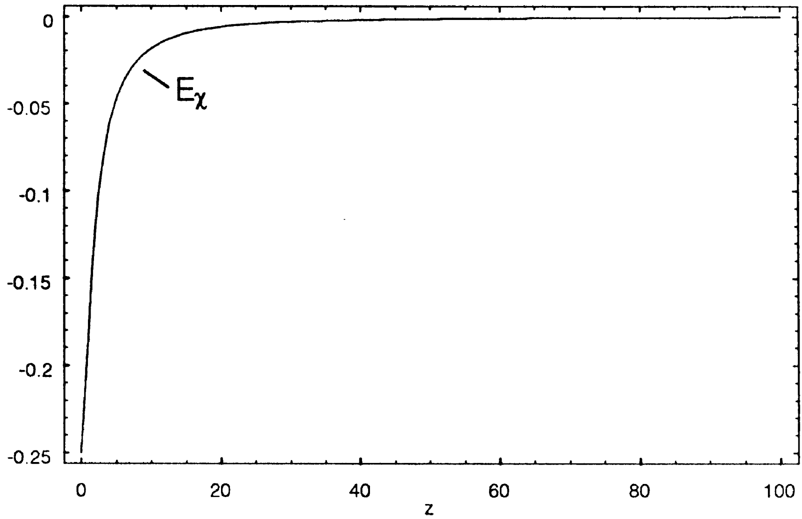

Fig. 4. Relative error E_χ as function of inverse temperature z.

One may extend the results of this section to the conclusion that the proper choice of moments and/or phase density depends on the interaction term of the relativistic Boltzmann equation.

6. Conclusions

The formalism of projected moments presents itself as a powerful tool in relativistic kinetic theory. Since the projected moments are the relativistic extension of the central moments in non-relativistic theory (Section 3.4) this new formalism fills a gap between relativistic and non-relativistic kinetic theory.

In particular, it allows an easy access to moment theories with more than 14 moments for the description of non-equilibrium processes via Extended Thermodynamics (Section 4). Here one may distinguish between isotropic non-equilibrium, which will be described only by scalar moments M_r, but with a wide range of numbers r, and non-isotropic non-equilibrium which will be described by tensorial moments $M_r^{A_1 \cdots A_n}$ with a wide range of values for n. The distinction between isotropic and non-isotropic non-equilibrium was already very useful in the case of relativistic radiative transfer, see [13]. This raises the question how many and which moments will be needed in order to describe a given process properly – the answer will hopefully be given in a future paper. But already the case of 14 moments needs further consideration since we have presented an alternative set of moment equations for the 14-field case (93) which competes with the old 14-field system (91).

A first answer to these questions was given in Section 5 of this paper where we considered the local equilibrium case. It turned out that the proper choice of moment equations depends on the collision term under consideration. If one goes from the BGK model to the relativistic Boltzmann collision term, the projected moment formalism will also be useful in order to calculate the relativistic Navier–Stokes and Fourier laws, for instance, with the combined method of Chapman–Enskog and Grad [20–22].

Appendix A. Motivation of relativistic BGK equation

The non-relativistic BGK equation reads [17]

$$\frac{\partial f}{\partial t} + \xi^i \frac{\partial f}{\partial x^i} = -\frac{1}{\tau}(f - f_{|E}),$$

where τ is the mean collision-free time. We use Eq. (7) to write

$$\frac{\partial f}{\partial t} + c\frac{p^i}{p^0}\frac{\partial f}{\partial x_i} = -\frac{1}{\tau}(f - f_{|E})$$

or, after multiplication with p^0/c

$$p^A f_{,A} = -\frac{p^0}{c\tau}(f - f_{|E}).$$

Since f is an invariant scalar, this equation is invariant with respect to Lorentz transformations only if $p^0/c\tau$ is an invariant scalar. Thus, either τ has to transform like the time component of a 4-vector or p^0 has to be taken in a fixed frame. We choose the second possibility and replace p^0 by p^0_{LL} (Landau–Lifshitz frame) on the right-hand side. Thus, follows Eq. (32). Note that the choice of the Landau–Lifshitz frame guarantees the conservation of particle number, energy and momentum.

Appendix B. Calculation of Lagrange multipliers

In this appendix we give some details for the calculation of the Lagrange multipliers in the 14 moment case.

We start with the calculation of the moments as functions of the Lagrange multipliers. Insertion of Eq. (117) into (107) for $n=0,1,2$ yields in the first step

$$\begin{aligned}
M_r &= -\lambda^{-1} M_{r-1\,|\,E} - (\lambda^0 - 1) M_{r\,|\,E} - \hat{\lambda}^1 M_{r+1\,|\,E} - \lambda^0_A M^A_{r\,|\,E} - \lambda^1_A M^A_{r+1\,|\,E}\,,\\
M^A_r &= -\lambda^{-1} M^A_{r-1\,|\,E} - (\lambda^0 - 1) M^A_{r\,|\,E} - \hat{\lambda}^1 M^A_{r+1\,|\,E}\\
&\quad - \lambda^0_B M^{AB}_{r\,|\,E} - \lambda^1_B M^{AB}_{r+1\,|\,E} - \lambda^1_{\langle BC\rangle} M^{A\langle BC\rangle}_{r+1\quad |E}\,,\\
M^{\langle AB\rangle}_r &= -\lambda^{-1} M^{\langle AB\rangle}_{r-1\,|\,E} - (\lambda^0 - 1) M^{\langle AB\rangle}_{r\,|\,E} - \hat{\lambda}^1 M^{\langle AB\rangle}_{r+1\,|\,E}\\
&\quad - \lambda^0_C M^{\langle AB\rangle C}_{r\,|\,E} - \lambda^1_C M^{\langle AB\rangle C}_{r+1\,|\,E} - \lambda^1_{\langle CD\rangle} M^{\langle AB\rangle\langle CD\rangle}_{r+1\quad |E}\,.
\end{aligned} \tag{B.1}$$

We introduce the equilibrium moments of the Landau–Lifshitz frame by Eq. (68) and neglect all terms of second order in the λ's and w^A to obtain

$$M_r = -\lambda^{-1} m_{r-1} - (\lambda^0 - 1) m_r - \hat{\lambda}^1 m_{r+1}\,, \tag{B.2a}$$

$$M_r^A = \left(m_r - \frac{r}{3} m_{r\,D}^{D} \right) \frac{1}{c} w^A - \lambda^{0A} \frac{1}{3} m_{r\,D}^{D} - \lambda^{1A} \frac{1}{3} m_{r+1\,D}^{D}\,, \tag{B.2b}$$

$$M_r^{\langle AB \rangle} = -\frac{2}{15} \lambda^{1\langle AB \rangle} m_{r+1\,FG}^{FG}\,. \tag{B.2c}$$

From these equation we shall now calculate the Lagrange multipliers as function of the moments. We start with the determination of the scalar Lagrange multipliers $\lambda^{-1}, \lambda^0, \hat{\lambda}^1$. These follow from Eq. (B.2a) with $r = 0, 1$

$$M_0 = -\lambda^{-1} m_{-1} - (\lambda^0 - 1) m_0 - \hat{\lambda}^1 m_1\,,$$

$$M_1 = -\lambda^{-1} m_0 - (\lambda^0 - 1) m_1 - \hat{\lambda}^1 m_2$$

and the equation for the pressure $p = -\frac{1}{3} M_{1\,A}^{A} = \frac{1}{3}(M_1 - m^2 c^2 M_{-1})$

$$-3p = M_{1\,A}^{A} = m^2 c^2 (-\lambda^{-1} m_{-2} - (\lambda^0 - 1) m_{-1} - \hat{\lambda}^1 m_0)$$

$$+ \lambda^{-1} m_0 + (\lambda^0 - 1) m_1 + \hat{\lambda}^1 m_2\,.$$

With Eqs. (71), (65a) and (65b) we may simplify these equations to

$$\begin{bmatrix} I_{-1} & I_0 & I_1 \\ I_0 & I_1 & I_2 \\ I_{-2} & I_{-1} & I_0 \end{bmatrix} \begin{bmatrix} \frac{\lambda^{-1}}{mc} \\ \lambda^0 \\ \hat{\lambda}^1 mc \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{3(p - p_{|E})}{mc M_0 / I_0} \end{bmatrix},$$

where $p_{|E} = \frac{1}{3}(m_1 - m^2 c^2 m_{-1})$ is the equilibrium pressure. The combination $p - p_{|E} = m^2 c^2 / 3 (m_{-1} - M_{-1})$ is the dynamical pressure. Inversion of the inhomogeneous system gives the scalar Lagrange multipliers as given in Eq. (118).

Eq. (B.2b) with $r = 0, 1$ serves to determine λ^{0A} and λ^{1A}. After elimination of w^A by means of Eq. (71) and with the use of Eqs. (65a) and (65b) these equations read

$$3 \frac{I_0}{M_0} \left(M_0^A - \frac{I_0}{mc(I_1 - \frac{1}{3}(I_{-1} - I_1))} M_1^A \right) = -\lambda^{0A}(I_{-2} - I_0) - mc \lambda^{1A} (I_{-1} - I_1)\,,$$

$$0 = -\lambda^{0A}(I_{-1} - I_1) - mc \lambda^{1A} (I_0 - I_2)\,.$$

Inversion gives Eq. (119).

Eq. (120) follows immediately from Eq. (B.2c) with $r = 1$.

Appendix C. Chernikov's phase density

We consider Chernikov's phase density (116)

$$f = f_{|E}(1 - \hat{\alpha}_A p^A - \hat{\beta}_{AB} p^A p^B)$$

and rewrite it as

$$f = f_{|E}(1 - \alpha^0 - \alpha^1 p^0_{|LR} - \alpha^2 (p^0_{|LR})^2 - \alpha^0_A R^A - \alpha^1_A p^0_{|LR} R^A - \alpha_{\langle AB\rangle} R^{\langle A} R^{B\rangle}) . \tag{C.1}$$

Here, the coefficients must be determined from the constraints

$$\begin{aligned} M_0 &= c \int p^0_{|LR} f \, dP, \qquad M^A_0 = c \int R^A f \, dP , \\ M_1 &= c \int (p^0_{|LR})^2 f \, dP, \qquad M^A_1 = 0 = c \int p^0_{|LR} R^A f \, dP , \\ M^{AB}_1 &= c \int R^A R^B f \, dP , \end{aligned} \tag{C.2}$$

where we have set $M^A_1 = 0$ which means that we are interested in the Landau–Lifshitz frame only.

From Eq. (C.1) we obtain the projected moments (48) for $n = 0, 1, 2$ as

$$M_r = (1 - \alpha^0) m_r - \alpha^1 m_{r+1} - \alpha^2 m_{r+2} , \tag{C.3a}$$

$$M^A_r = -\tfrac{1}{3}\alpha^{0A}((mc)^2 m_{r-1} - m_{r+1}) - \tfrac{1}{3}\alpha^{1A}((mc)^2 m_r - m_{r+2}) , \tag{C.3b}$$

$$M^{\langle AB\rangle}_r = -\tfrac{2}{15}\alpha^{\langle AB\rangle} m^{FG}_{r+2FG} . \tag{C.3c}$$

From these equations we shall now calculate the coefficients as functions of the moments. We start with the determination of the scalar Lagrange multipliers $\alpha^0, \alpha^1, \alpha^2$. These follow from Eq. (C.3a)

$$\begin{aligned} M_0 &= (1 - \alpha^0) M_0 - \alpha^1 M_1 - \alpha^2 m_2 , \\ M_1 &= (1 - \alpha^0) M_1 - \alpha^1 m_2 - \alpha^2 m_3 \end{aligned}$$

and the equation for the pressure

$$\begin{aligned} 3(p - p_{|E}) &= -M^A_{1\,A} + m^A_{1\,A} = (mc)^2 (m_{-1} - M_{-1}) \\ &= (mc)^2 (\alpha^0 m_{-1} + \alpha^1 M_0 + \alpha^2 M_1) . \end{aligned}$$

With Eq. (74) we may simplify these equations to

$$\begin{bmatrix} I_0 & I_1 & I_2 \\ I_1 & I_2 & I_3 \\ I_{-1} & I_0 & I_1 \end{bmatrix} \begin{bmatrix} \alpha^0 \\ \alpha^1 mc \\ \alpha^2 (mc)^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{3(p - p_{|E})}{mc(M_0/I_0)} \end{bmatrix}$$

and obtain by inversion the coefficients $\alpha^0, \alpha^1, \alpha^2$

$$\alpha^0 = \frac{I_1 I_3 - I_2^2}{I_{-1}I_1I_3 + 2I_0I_1I_2 - I_{-1}I_2^2 - I_0^2 I_3 - I_1^3} \frac{3(p - p_{|E})}{mc(M_0/I_0)} ,$$
$$\alpha^1 mc = \frac{I_1 I_2 - I_3 I_0}{I_{-1}I_1I_3 + 2I_0I_1I_2 - I_{-1}I_2^2 - I_0^2 I_3 - I_1^3} \frac{3(p - p_{|E})}{mc(M_0/I_0)} , \qquad (C.4)$$
$$\alpha^2 (mc)^2 = \frac{I_0 I_2 - I_1^2}{I_{-1}I_1I_3 + 2I_0I_1I_2 - I_{-1}I_2^2 - I_0^2 I_3 - I_1^3} \frac{3(p - p_{|E})}{mc(M_0/I_0)} .$$

Eq. (C.3b) with $r = 0, 1$ serves to determine α^{0A} and α^{1A},

$$M_0^A = -\tfrac{1}{3}\alpha^{0A}((mc)^2 m_{-1} - M_1) - \tfrac{1}{3}\alpha^{1A}((mc)^2 M_0 - m_2) ,$$
$$0 = -\tfrac{1}{3}\alpha^{0A}((mc)^2 M_0 - m_2) - \tfrac{1}{3}\alpha^{1A}((mc)^2 M_1 - m_3) .$$

After use of Eq. (74) and some algebra we obtain

$$\alpha^{0A} = \frac{I_1 - I_3}{(I_0 - I_2)^2 - (I_{-1} - I_1)(I_1 - I_3)} \frac{M_0^A}{\frac{1}{3}mc(M_0/I_0)} ,$$
$$mc\alpha^{1A} = \frac{I_2 - I_0}{(I_0 - I_2)^2 - (I_{-1} - I_1)(I_1 - I_3)} \frac{M_0^A}{\frac{1}{3}mc(M_0/I_0)} . \qquad (C.5)$$

Eq. (C.3c) with $r = 1$ gives immediately the coefficient $\alpha^{\langle AB \rangle}$ as

$$\alpha^{\langle AB \rangle} = -\frac{15}{2} \frac{1}{I_{-1} - 2I_1 + I_3} \frac{M_1^{\langle AB \rangle}}{(mc)^3 (M_0/I_0)} . \qquad (C.6)$$

With (C.4)–(C.6) we know all coefficients in the phase density (C.1) and are able to calculate constitutive equations for arbitrary moments of this phase density. From easy calculations we obtain

$$M_r - m_r = -3(p - p_{|E})(mc)^{r-1} \times \frac{(I_1I_3 - I_2^2)I_r + (I_1I_2 - I_0I_3)I_{r+1} + (I_0I_2 - I_1^2)I_{r+2}}{I_{-1}I_1I_3 + 2I_0I_1I_2 - I_{-1}I_2^2 - I_0^2 I_3 - I_1^3} , \qquad (C.7)$$

$$\frac{M_r^A}{(mc)^r} = \frac{(I_{r-1} - I_{r+1})(I_1 - I_3) - (I_r - I_{r+2})(I_0 - I_2)}{(I_1 - I_3)(I_{-1} - I_1) - (I_0 - I_2)^2} M_0^A , \qquad (C.8)$$

$$M_r^{\langle AB \rangle} = (mc)^{r-1} \frac{I_{r-2} - 2I_r + I_{r+2}}{I_{-1} - 2I_1 + I_3} M_1^{\langle AB \rangle} . \qquad (C.9)$$

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