Levermore-Eddington factor and entropy maximization

Henning Struchtrup

Technical University Berlin, Sekr. HF 2, Fasanenstr. 90, 10623 Berlin, Germany

e-mail: hest0931@thermodynamik.tu-berlin.de

ABSTRACT

We discuss the physical significance of a special Eddington factor, introduced into radiative transfer theory by Levermore (1983) and obtained also by Anile et.al. (1991) and Kremer&Müller (1992). Since that Eddington factor follows from the maximization of entropy, it is not suitable for the description of radiation beams. Indeed, there are only few physical situations in which it may play a role.

PACS 95.30.Jx/42.86.Ay/44.40.+a (Radiative Transfer)

1 EDDINGTON FACTOR

The balances of radiative energy density e and radiative momentum density p_i may be written as

$$\frac{\partial e}{\partial t}+\frac{\partial Q_j}{\partial x_j}=P_e\,,\qquad \frac{\partial p_i}{\partial t}+\frac{\partial N_{ij}}{\partial x_j}=P_{p_i}\,. \tag{1}$$

Here Q_i denotes the radiative energy flux, N_{ij} is the radiative pressure tensor and P_e, P_{p_i} stand for the interchange of energy and momentum with matter, due to absorption, emission and scattering.

We describe the radiation in the photon picture. The energy of a photon is $\hbar\omega$ and its momentum is given by $\hbar\mathbf{k}$. Frequency ω and wave vector $\mathbf{k}$ are related by the dispersion relation $\omega = ck = c\,|\mathbf{k}|$; c is the speed of light and $\hbar$ denotes Planck's constant divided by 2π. The direction of propagation is given by $\mathbf{k}/k$.

The phase density f is defined such that

$$f\left(\mathbf{x},\mathbf{k},t\right)d\mathbf{k}=f\left(\mathbf{x},\mathbf{k},t\right)k^2\,dk\,d\Omega \tag{2}$$

gives the number density of photons with wave vectors between $\mathbf{k}$ and $\mathbf{k}+d\mathbf{k}$; $d\Omega$ is the solid angle element $\sin\vartheta\,d\vartheta\,d\varphi$ in the direction (ϑ,φ) of propagation. With these definitions the densities of energy and momentum of the photon gas and their fluxes are given by the relations [1, 2]:

$$e=\int\hbar ck f\,d\mathbf{k} \tag{3}$$

$$p_i=\frac{1}{c^2}Q_i=\int\hbar k_i f\,d\mathbf{k}, \tag{4}$$

$$N_{ij}=\int\hbar c\frac{k_i k_j}{k}f\,d\mathbf{k}\,. \tag{5}$$

From (3), (5) follows that the trace of the pressure tensor is equal to the energy density,

$$N_{ii}=\int\hbar ck f\,d\mathbf{k}=e\,. \tag{6}$$

The equations (1) do not form a closed set of equations for the radiative quantities e and p_i, since they include the unknown pressure tensor N_{ij} and the productions terms P_e and P_{p_i}. The latter can only be calculated by detailed knowledge of the interaction of radiation and matter and we do not consider them in the sequel. Here we are interested in the pressure tensor N_{ij} alone.

In general N_{ij} will depend on space and time via the actual values of e and p_i and their space-time derivatives of any order. Under the special assumption that the pressure tensor is independent of all gradients, i.e. depends only on e and p_i this dependence must be of the form

$$N_{ik}=e\left(\frac{1-\chi}{2}\delta_{ik}+\frac{3\chi-1}{2}\frac{p_i p_k}{p^2}\right) \tag{7}$$

where χ depends on e and $p^2=p_i p_i$. Obviously (7) fulfills the trace condition (6). The function $\chi\left(e,p^2\right)$ is called Eddington factor and it remains to find its explicit form.

By the simple assumption that there is a special frame, in which the radiation field is isotropic, Levermore [3] obtained the Eddington factor as

$$\chi_L=\frac{5}{3}-\frac{4}{3}\sqrt{1-\frac{3}{4}\frac{c^2p^2}{e^2}}. \tag{8}$$

Anile et.al. [4] have derived the same function for the Eddington factor by extended thermodynamics and so did Kremer&Müller [5] in a slightly different manner.

For the equilibrium case $p_i = 0$ the Levermore-Eddington factor becomes $\chi_L=\frac{1}{3}$. For $p_i = e/cn_i$, when all photons travel in the same direction n_i, we have $\chi_L = 1$ and find

$$N_{ij}=en_i n_j. \tag{9}$$

Thus it would seem that the theory is able to cover the whole range of states from equilibrium to the free streaming case, i.e. beams of radiation.

The aim of this paper is to discuss, in which physical situations the Levermore-Eddington factor is appropriate. In particular, we shall show that the Eddington factor (8) is not suitable for the decription of radiation beams, because it implies an infinite energy density for the beam. Moreover, we shall present arguments to prove that this Eddington factor pertains to the case of radiation in equilibrium with moving matter, at least in the most cases. We will also discuss situations in which the Levermore-Eddington factor is suitable for non-equilibrium cases.

The entropy maximum principle [6] is equivalent to extended thermodynamics and we will use it to derive the equilibrium phase density in Section II and a phase density which will lead to the Levermore-Eddington factor in Section III. In section IV we shall compare it with the equilibrium phase density seen from outside the rest frame of matter. Thus we will come to the above mentioned conclusions.

2 EQUILIBRIUM

Equilibrium of a photon gas means that it is in equilibrium with matter of temperature T. The energy density e characterizes the equilibrium state of the photon gas completely and we may obtain the equilibrium phase density $f_{|E}$ by maximization of the radiative entropy under the constraint of a prescribed value for the energy density $e = \int \hbar c k f \, d\mathbf{k}$. The radiative entropy density is given by [7, 8]

$$h = -k_B \int \left[f \ln \frac{f}{y} - (y+f) \ln \left(1 + \frac{f}{y} \right) \right] d\mathbf{k} \tag{10}$$

where $y = 2/(2\pi)^3$ and k_B denotes Boltzmann's constant. We take care of the constraint by a Lagrangian multiplier Λ_e and obtain

$$f_{|E} = \frac{y}{\exp \frac{\hbar c k}{k_B} \Lambda_e - 1}. \tag{11}$$

It remains to calculate the Lagrange multiplier Λ_e from the constraint and we find

$$\Lambda_e = \left(\frac{a}{e} \right)^{1/4} \tag{12}$$

where $a = \frac{\pi^2}{15} \frac{k_B^4}{\hbar^3 c^3}$ denotes the radiation constant. In equilibrium the Stephan-Boltzmann law $e = aT^4$ holds and we indentify $\Lambda_e = 1/T$. With (11) we obtain from (5) for the pressure tensor

$$N_{ij} = \frac{e}{3} \delta_{ij} \tag{13}$$

Comparison with the definition of the Eddington factor (7) shows that $\chi = 1/3$ in this case.

Now we ask for the phase densitiy and the fields which are observed in a Lorentz frame that moves with the velocity v_i relative to the rest frame of matter. Let $\hat{k}$ and $\hat{n}_i$ denote wave number and direction vector in this frame. Then we have, due to the Doppler effect, the following relation between the wave number k in the rest frame and $\hat{k}$ and $\hat{n}_i$ [9]

$$k = \frac{\hat{k}}{\sqrt{1 - \frac{v^2}{c^2}}} \left(1 - \frac{1}{c} v_k \hat{n}_k \right) \tag{14}$$

Since the phase density is a relativistic invariant [1] the distribution function in the moving frame is given by

$$\hat{f} = \frac{y}{\exp \left[\frac{\hbar c \hat{k}}{k_B T} \frac{\left(1 - \frac{1}{c} v_k \hat{n}_k \right)}{\sqrt{1 - \frac{v^2}{c^2}}} \right] - 1}. \tag{15}$$

The value of the fields follows from (3), if we replace k,n_i, f by $\hat{k}$,$\hat{n}_k$, $\hat{f}$. We obtain

$$e = aT^4 \frac{1 + \frac{1}{3}\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}, \quad p_i = \frac{4}{3} \frac{aT^4}{c^2} \frac{v_i}{1 - \frac{v^2}{c^2}}, \tag{16}$$

$$N_{ij} = aT^4 \left(\frac{1}{3} \delta_{ij} + \frac{4}{3} \frac{\frac{1}{c^2} v_i v_j}{1 - \frac{v^2}{c^2}} \right), \tag{17}$$

for the Eddington factor we find

$$\hat{\chi} = \frac{1 + 3\frac{v^2}{c^2}}{3 + \frac{v^2}{c^2}}. \tag{18}$$

The anisotropy is due to aberration. If v approaches the speed of light c we have $p_i = e/c \, n_i$ with the unit direction vector $n_i = v_i/c$. This is the condition of the free streaming case, when all photons move in the same direction n_i; note that both, e and p_i, tend to infinity in this case.

3 LEVERMORE EDDINGTON FACTOR

Now we assume, that e and p_i characterize the state of the photon gas completely. To determine the phase density f_p for this case, we maximize the radiative entropy (10) under the constraint of prescribed values for e and p_i. Again we take care of the constraints by Lagrangian multipliers Λ_e, Λ_i and find

$$f_p = \frac{y}{\exp \left[\frac{\hbar c k}{k_B} \left(\Lambda_e + \Lambda_i n_i \right) \right] - 1} \tag{19}$$

From (3), (4) we obtain with (19)

$$\begin{aligned} e &= \int \hbar c k f_p \, d\mathbf{k} = \frac{a}{3} \frac{3 (\Lambda_e)^2 + \Lambda_i \Lambda_i}{\left[(\Lambda_e)^2 - \Lambda_i \Lambda_i \right]^3} \\ p_i &= \int \hbar k_i f_p d\mathbf{k} = -\frac{1}{c} \frac{4a}{3} \frac{\Lambda_e}{\left[(\Lambda_e)^2 - \Lambda_i \Lambda_i \right]^3} \Lambda_i \end{aligned} \tag{20}$$

By inversion we obtain Λ_e and Λ_i in terms of e and p_i,

$$\Lambda_e = \left(\frac{a}{e}\right)^{1/4} \frac{(3-\chi_L)^{1/2}}{2^{3/4}3^{1/4}(1-\chi_L)^{3/4}}$$
$$\Lambda_i = -\left(\frac{a}{e}\right)^{1/4} \frac{2^{1/4}\frac{cp_i}{e}}{3^{1/4}(1-\chi_L)^{3/4}(3-\chi_L)^{1/2}} \tag{21}$$

where we have introduced the Levermore Eddington factor $\chi_L = \frac{5}{3} - \frac{4}{3}\sqrt{1-\frac{3}{4}\frac{c^2p^2}{e^2}}$ (8) for abbreviation. To our knowledge these calculations were first done by Larecki [10] in the field of the kinetic theory of phonons.

The phase density f_p is thus explicitly related to e and p_i and we may now calculate the radiative pressure tensor N_{ij} from (5). We obtain

$$N_{ij} = e\left(\frac{1-\chi_L}{2}\delta_{ij} + \frac{3\chi_L - 1}{2}\frac{p_i p_j}{p^2}\right)$$

which is (7) with the Levermore Eddington factor. The above calculations rely on the entropy maximum principle which is equivalent to extended thermodynamics [6]. Thus this procedure is equivalent, although easier, to the ones by Anile et al. [4] and Kremer&Müller [5]; and naturally we obtain the same results. The entropy maximum principle gives not only the Eddington factor, but also the corresponding phase density.

By comparison of (19) and (15) we find that we can replace the unknowns Λ_e, Λ_i in (19) by two other unknowns T, v_i such that

$$\Lambda_e = \frac{1}{T}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad \Lambda_i = -\frac{1}{T}\frac{v_i/c}{\sqrt{1-\frac{v^2}{c^2}}} \tag{22}$$

and therefore we can write (19) as

$$f_p = \frac{y}{\exp\left[\frac{\hbar ck}{k_B T}\frac{\left(1-\frac{1}{c}v_k n_k\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right] - 1}. \tag{23}$$

From (21), (22) we find for the relations between e, p_i and T and v_i

$$T = \left(\frac{e}{a}\right)^{1/4} \frac{3^{1/4}}{2^{1/4}}(1-\chi_L)^{1/4} \tag{24}$$
$$\frac{v_i}{c} = \frac{2}{(3-\chi_L)}\frac{cp_i}{e} \tag{25}$$

We conclude from (23) that there exists a Lorentz frame - moving with the speed v_i given in (25) - where the phase density is isotropic. We write the phase density in this special frame as

$$\tilde{f}_p = \frac{y}{\exp\frac{\hbar c\tilde{k}}{k_B T} - 1}, \tag{26}$$

where $\tilde{k}$ is the wave number in the new frame

$$\tilde{k} = \frac{k}{\sqrt{1-\frac{v^2}{c^2}}}\left(1 - \frac{1}{c}v_k n_k\right). \tag{27}$$

$\tilde{f}_p$ in (26) is the equilibrium phase density with temperature T as given in (24).

In other words: If the radiation field is characterized by the phase density (19) - with Λ_e, Λ_{p_i} as in (21) - there exists a frame where this radiation field has the equilibrium phase density appropriate to the temperature T and this frame moves with velocity v_i. T and v_i are related to e and p_i by (24), (25).

4 INTERPRETATION

In the limit $\chi_L \rightarrow 1$, which is supposed to correspond to beams, we find from (25) that this special frame moves with the speed of light. From (24) we have to conclude that the energy density of the beam is infinite, since the temperature T in the rest frame must be unequal to zero. It follows that the Levermore-Eddington factor should not be used for the description of beams.

Note that the frequency distribution of beams emerging from the surfaces of stars obeys Planck's law, but the distribution is anisotropic (see Section 4 of Struchtrup 1997 [11]). There is no transformation which will lead to an isotropic distribution.

The only entropy-maximizing process which will lead to the equilibrium phase density is the interaction of radiation with matter of temperature T at rest. We have to conclude that the Levermore-Eddington factor belongs to physical situations where the radiation *is* in equilibrium with matter of temperature T, or at least *has been* in equilibrium with such matter.

In the first case, where the radiation *is* in equilibrium with matter, the Eddington factor is equal to $\frac{1}{3}$ in the rest frame of the matter. If the radiation is not observed in the rest frame but in a moving frame, the Eddington factor has the value $\chi_L = \frac{5}{3} - \frac{4}{3}\sqrt{1-\frac{3}{4}\frac{c^2p^2}{e^2}}$; this complicated form of the Eddington factor results entirely from the motion of the matter relative to the observer. The phase density (19) with (21) does not characterize a non-equilibrium between matter and radiation, - or only apparently.

In the second case the radiation *has been* in equilibrium with matter of temperature T. We may ask for cases where at one time equilibrium prevailed between radiation and matter and does no more. Actually equilibrium may be lost by the "decoupling" of matter and radiation. After decoupling the matter may be removed leaving isotropic radiation. After that new matter - which moves relative to the frame of the isotropic radiation - may be inserted in the radiation field. Thus one ends up with an isotropic Planck law for radiation outside the rest frame of matter.

There remains the question of how radiation and matter may be decoupled. The best-known case is the cosmic background radiation, where the decoupling is due to the expansion of the universe [12]. Another possible way to achieve equilibrium radiation without matter, is to consider a cylinder with ideally reflecting walls. First one has a small piece of matter, which absorbs and emits all frequencies, of temperature T in the cylinder so that the radiation is in equilibrium. Then, one removes the matter and obtains equilibrium radiation of temperature T in the cylinder, but no matter. Here, the decoupling process is the removal of the matter. Afterwards new matter maybe put into the cylinder. If the matter moves relative to the cylinder, we should have a proper non-equilibrium setting for the use of the Levermore Eddington factor. This setting is rather artificial and we cannot think of a real process which will lead to the same physical state, except the expansion of the universe.

Moreover, the description of radiation by the two quantities energy density e and momentum density p_i will in most cases not be in accordance with the radiative transfer equation. In two papers we have shown that already simple processes, such as one-dimensional beams or isotropic compression of radiation, require a large number of moments in order to give valid solutions[11, 13].

Acknowledgement: I am indebted to Prof. M. Anile (Catania) and Prof. I. Müller (Berlin) for valuable discussions.

REFERENCES

[1] D. Mihalas, B. W. Mihalas *Foundations of Radiation Hydrodynamics.* Oxford University Press, New York (1984)

[2] J. Oxenius *Kinetic Theory of Particles and Photons.* Springer, Berlin (1986)

[3] C.D. Levermore *Relating Eddington Factors to Flux Limiters.* J. Quant. Spectrosc. Radiat. Transfer 31 (1983)

[4] A.M. Anile, S. Pennisi, M. Sammartino *A Thermodynamical Approach to Eddington Factors.* J. Math. Phys. **32** (1991)

[5] G.M. Kremer, I. Müller *Radiation Thermodynamics.* J. Math. Phys. **33** (1992).

[6] W. Dreyer *Maximisation of the Entropy in Non-equilibrium.* J. Phys. A: Math.Gen. **20** (1987)

[7] A. Ore *Entropy of Radiation.* Phys. Rev **98** (1955)

[8] P. Rosen *Entropy of Radiation.* Phys. Rev **96** (1954)

[9] R. Becker, F. Sauter *Theorie der Elektrizität, Bd. 1*, Teubner Verl. Ges. Stuttgart (1957)

[10] W. Larecki *Symmetric Conservative Form of Low-temperature Phonon Gas Hydrodynamics II. Equations of Heat Transport and Thermal Waves in the Case of Linear Isotropic Approximation of Phonon Frequency Spectrum.* Il Nuovo Cimento **1868** D (1991)

[11] H. Struchtrup *On the Number of Moments in Radiative Transfer Problems.* (submitted)

[12] T. Fließbach *Allgemeine Relativitätstheorie*, B.I. Wissenschaftsverlag Mannheim (1990)

[13] H. Struchtrup *An Extended Moment Method in Radiative Transfer: The Matrices of Mean Absorption and Scattering Coefficients.* to appear in Annals of Physics (1997)