

An Extended Moment Method in Radiative Transfer: The Matrices of Mean Absorption and Scattering Coefficients

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In extension of the ideas of Anderson and Spiegel (1972) the radiative transfer equation is replaced by moment equations for the moments

$$A_r^{A_1 A_2 \dots A_N} = \int (p_{RL}^0)^{r+1-N} p^{A_1} p^{A_2} \dots p^{A_N} f dP,$$

$r=0, 1, \dots, R$. Here p^A is the photon 4-momentum, cp_{RL}^0 is the photon energy in the rest Lorentz frame and f is the photon phase density. From these follow moment equations for the projected symmetric trace free moments introduced by Thorne (1981). The required closure of the equations is achieved by use of a series expansion of the phase density which is motivated by the entropy maximum principle. This procedure provides a coupling of the moment equations by means of *matrices* of mean absorption and scattering coefficients. It is shown that the extension from $r=1$ (Anderson and Spiegel, 1972; Thorne, 1981; Schweizer, 1988) to $r=0, 1, \dots, R$ gives reasonable results: In the limit of local radiative equilibrium (LRE) the well-known Rosseland mean of the absorption coefficient is recovered. For a simple non-LRE experiment, the homogeneous compression and relaxation of radiation, the radiative transfer equation, and the moment equations are solved. The comparison of the results in the case of pure bremsstrahlung (free-free) absorption shows an excellent agreement for $R \geq 6$.

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1. INTRODUCTION

The goal of radiation thermodynamics is the determination of the transfer of radiative energy and momentum as well as of the interchange of energy and momentum between radiation and matter.

In principle the problem is solved when the radiative transfer equation with all interaction terms is known. Of interest are the thermodynamic quantities energy, momentum, energy and momentum interchange, entropy, etc. They may be calculated from the photon phase density or the intensity of radiation and the spectral absorption and scattering coefficients by integration over all photon frequencies and directions. The photon phase density follows as the solution of the radiative transfer equation which is—in general—not analytically solvable.

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An alternative approach to the problem is the method of moments: One derives moment equations from the radiative transfer equation and obtains an infinite set of partial differential equations which may replace the radiative transfer equation. Since it makes no sense to deal with an infinite number of equations one has to restrict one's attention to a limited number of moments. The question is how many and which moments are needed for a satisfactory description of radiative processes.

The restriction to a limited number of equations leads to another question: The finite set of moment equations contains more unknowns than variables. Therefore one needs constitutive equations to close the system. How can these be found?

There exists a great number of theories with different answers for both questions. Most authors consider only the moment equations for energy and momentum and close them by several methods. This choice gives excellent results in the case of local radiative equilibrium (LRE) where the photon phase density is a local Planck function, see [4, 5, 9]. Similar theories for non-LRE, e.g. [2, 10, 11, 12], however, do not provide a proper description of radiative processes because the variety of deviations from the equilibrium state cannot be described by the two quantities energy and momentum alone.

For this reason Anderson and Spiegel [1] develop a theory, in covariant formulation, which takes an arbitrary number of moments into account but they provide no method for the closure of their equations. This was done later by Thorne [22] and Schweizer [20] who took great care in the calculation of mean absorption and scattering coefficients. This theory does not seem altogether satisfactory because Anderson and Spiegel's choice of moments reflects the anisotropy of the non-equilibrium phase density only but not its spectral deviation from Planck's equilibrium phase density.

Therefore in the present paper we choose a more extended set of moments as variables. The equations are closed by means of an series representation of the phase density which is motivated by the entropy maximum principle of extended thermodynamics [7, 15]. Thus the phase density is a series in spherical harmonics, allowing for anisotropy, and powers of the frequency, allowing for spectral deviations. The expansion coefficients are related to the moments.

We obtain a set of partial differential equations which are coupled by *matrices* of mean absorption and scattering coefficients. In the limit of LRE these equations reproduce the well-known Rosseland mean of the absorption coefficient. By a simple experiment, the homogeneous compression of radiation, we show that the equations provide excellent results in non-LRE too. We obtain them by solving the radiative transfer equation as well as the moment equations and compare the results. A discussion of the equations for inhomogeneous problems will be published later.

The equations are presented in covariant form using the formalism of projected symmetric trace free (PSTF) moments which was introduced by Thorne [22]. We shall also consider the case of non-relativistic matter velocities in the laboratory frame and of matter at rest.

We are not interested here in discussing the importance of the various absorption and scattering effects at different temperatures. For this we refer, again, to the paper of Thorne [22]. We shall formulate the theory for arbitrary absorption and scattering coefficients; in the examples, however we will consider only bremsstrahlung (free-free) absorption and Thomson scattering at electrons.

2. EQUATIONS FOR MATTER

We describe the matter background as a one component gas in local thermal equilibrium (LTE) i.e. the phase density of the gas is a local Maxwellian. For reasons of simplicity we assume that all effects of viscosity and heat transfer in the gas can be neglected. For more precise theories the interested reader is referred to the literature [8, 15, 16].

We denote the 4-velocity of the matter by U^A . For an observer at rest in the accompanying rest Lorentz frame (RLF) we have

$$U^A = \{c, 0, 0, 0\}^A; \quad (2.1)$$

where c is the velocity flight. We choose the Lorentz metric as $\{1, -1, -1, -1\}$ so that

$$U^A U_A = c^2. \quad (2.2)$$

The observer in RLF measures the particle number density n_{RL} . The conservation of particle number density reads

$$(n_{RL} U^A)_{;A} = 0, \quad (2.3)$$

where the semicolon denotes covariant differentiation.

The covariant balance of energy momentum T^{AB} reads

$$T^{AB}_{;B} = -P^A, \quad (2.4)$$

where the 4-vector $-P^A$ accounts for the exchange of energy and momentum with radiation. The energy momentum tensor T^{AB} may be decomposed as [16]

$$T^{AB} = \varepsilon_{RL} \frac{1}{c^2} U^A U^B - p \Delta^{AB}, \quad (2.5)$$

where

$$\varepsilon_{RL} = n_{RL} m c^2 + n_{RL} u \quad (2.6)$$

is the energy density in RLF, m is the particle rest mass and u is the thermal energy density. Moreover p is the pressure in RLF and the spatial projector is defined as

$$\Delta^{AB} = g^{AB} - \frac{1}{c^2} U^A U^B$$

with

$$\Delta^{AB} U_B = 0, \quad \Delta^{AB} \Delta_{BC} = \Delta^A{}_C, \quad \Delta^A{}_A = 3 \quad (2.7)$$

where g^{AB} denotes the metric tensor.

Supplemented by equations of state for $p = p(n_{RL}, T)$ and $u = u(n_{RL}, T)$ Eqs. (2.3, 2.4) provide five equations for the five variables temperature T , particle number density n_{RL} and velocity U^A . The temperature is a scalar.

The production vector P^A must be determined from radiation thermodynamics.

3. MOMENTS AND MOMENT EQUATIONS

3.1. Photons

We describe radiation in the photon picture. The photon 4-momentum of photons with frequency ω and direction vector n^i is given by

$$p^A = \frac{\hbar\omega}{c} \{1, n^i\}^A. \quad (3.1)$$

$\hbar = h/2\pi$ is Planck's constant, $\hbar\omega$ is the photon energy and $(\hbar\omega/c) n^i$ is the photon momentum. We have

$$p^A p_A = 0. \quad (3.2)$$

For the definition of the PSTF moments we need a decomposition of p^A into one part parallel to U^A and one part perpendicular to U^A [22],

$$p^A = p_{RL}^0 \left(\frac{1}{c} U^A + L^A \right) \quad \text{with} \quad L^A U_A = 0, \quad p_{RL}^0 = \frac{p^A U_A}{c}. \quad (3.3)$$

Because of

$$p_{RL}^A = p_{RL}^0 \{1, L^i_{RL}\}^A = \frac{\hbar\omega_{RL}}{c} \{1, n^i_{RL}\}^A, \quad (3.4)$$

cp_{LR}^0 is the photon energy in RLF and L^A is the covariant generalization of the photon direction vector in RLF with

$$L^A L_A = -1, \quad p^A \Delta_A{}^C = p_{RL}^0 L^C \quad (3.5)$$

In this paper we consider only unpolarized radiation so that we do not need any other quantities for the description of radiation.

In the following, for simplicity, we will denote frequency and direction vector in RLF simply by ω, n^i .

3.2. Photon Phase Density

The photon phase density $f(x^A, p_i)$ is defined such that

$$f(x^A, p_i) dX dP \quad (3.6)$$

gives the number of photons in the element $dX dP = \sqrt{-g} p^0 d^3x dP$ [6, 13]. Due to this definition f is an invariant scalar. dP denotes the invariant momentum space element

$$dP = \frac{\sqrt{-g}}{p_0} d^3p \quad \text{with} \quad dP_{RL} = \left(\frac{h}{c}\right)^2 \omega d\omega d\Omega,$$

and

$$d\Omega = \sin \vartheta d\vartheta d\varphi. \quad (3.7)$$

The energy momentum tensor of radiation is given by

$$T^{AB} = c \int p^A p^B f dP. \quad (3.8)$$

Later we will need the entropy 4-vector, given by [8, 18]

$$S^A = -k_B c \int p^A \left[f \ln \frac{f}{y} - (y + f) \ln \left(1 + \frac{f}{y} \right) \right] dP \quad \text{with} \quad y = \frac{2}{(2\pi h)^3}, \quad (3.9)$$

k_B is Boltzmann's constant.

3.3. Radiative Transfer Equations

The radiative transfer equation for f is equal to the covariant Boltzmann equation, viz. [6, 13]

$$p^B f_{;B} - \Gamma_{BC}^k p^B p^C \frac{\partial f}{\partial p^k} = \frac{p_{RL}^0}{c} \mathcal{S} = \frac{p_{RL}^0}{c} (\mathcal{S}_{A/E} + \mathcal{S}_{Sc}) \quad (3.10)$$

where Γ_{BC}^A are Christoffel symbols. \mathcal{S} denotes the invariant interaction term which is composed of one part $\mathcal{S}_{A/E}$ describing absorption and emission processes and of one part \mathcal{S}_{Sc} describing scattering processes. We will evaluate \mathcal{S} always in the RLF

so it is sufficient to give its form only in terms of RLF quantities. For absorption/emission processes we have, since the gas is in LTE [4]

$$\mathcal{G}_{A|E} = -\kappa(\omega)(f - f_{|E}). \quad (3.11)$$

$f_{|E}$ denotes the equilibrium phase density

$$f_{|E} = \frac{y}{\exp \Sigma_{|E} - 1}, \quad \text{with} \quad \Sigma_{|E} = \frac{cp_{RL}^0}{k_B T} = \frac{h\omega}{k_B T} \quad (3.12)$$

and κ is the effective spectral absorption coefficient. In the case of pure bremsstrahlung absorption we have [3, 4]

$$\kappa_{ff} = D_{ff} \frac{1 - \exp(-\Sigma_{|E})}{\Sigma_{|E}^3}, \quad D_{ff} = \varrho n_{el} \left(\frac{k_B T}{h} \right)^{-3} B(Z, n). \quad (3.13)$$

The constant $B(Z, n)$ depends on atomic constants, ϱ is the density of matter and n_{el} is the number density of free electrons.

The interaction term for anisotropic scattering reads [22]

$$\begin{aligned} \mathcal{L}_{Sc} = & -\zeta(\omega) \left[f(\omega, n^i) - \frac{1}{4\pi} \int f(\omega, n'^i) d\Omega' \right. \\ & \left. - \frac{3}{16\pi} n^{<i} n'^j \int n'^{<i} n'^j f(\omega, n'^i) d\Omega' \right] \end{aligned} \quad (3.14)$$

with the scattering coefficient ζ . The indices in angular brackets denote asymmetric trace-free tensor in 3-space (see below). For simplicity we may restrict the attention to isotropic scattering with

$$\mathcal{L}_{Sc} = -\zeta(\omega) \left[f(\omega, n^i) - \frac{1}{4\pi} \int f(\omega, n'^i) d\Omega' \right]. \quad (3.15)$$

In the case of Thomson scattering at electrons the scattering coefficient is independent of frequency,

$$\zeta_T = n_{el} \text{const.} \quad (3.16)$$

3.4. Moments

3.4.1. *Unprojected moments.* Extending the ideas of Anderson and Spiegel [1] we define unprojected moments as

$$A_r^{A_1 A_2 \dots A_N} = \int (p_{RL}^0)^{r+1-N} p^{A_1} p^{A_2} \dots p^{A_N} dP, \quad r, N \in \mathbb{N} \quad (3.17)$$

These are symmetric trace-free 4-tensors with, see (3.2, 3.3),

$$A_r^{A_1 A_2 \dots A_{N-2} A_{N-1}}{}_{A_{N-1}} = 0, \quad A_r^{A_1 A_2 \dots A_N} \frac{1}{c} U_{A_N} = A_r^{A_1 A_2 \dots A_{N-1}}. \quad (3.18)$$

The energy momentum tensor (3.8) is related to these moments by

$$T^{AB} = c A_1^{AB}. \quad (3.19)$$

Because of (3.18)₂ it is contained in the moments $A_1^{A_1 A_2 \dots A_N}$ ($N \geq 2$).

Note that Anderson and Spiegel choose only the moments with the index $r = 1$ while they left N arbitrary. The extension to other values of r is the main contribution of this paper.

3.4.2. PS moments. Extending the ideas of Thorne [22] we define projected symmetric (PS) moments as

$$M_r^{A_1 \dots A_n} = \int (p_{RL}^0)^{r+1} L^{A_1} \dots L^{A_n} f dP, \quad r, n \in \mathbb{N}. \quad (3.20)$$

The $M_r^{A_1 \dots A_n}$ have the following properties, see (3.3, 3.5):

$$M_r^{A_1 \dots A_n} U_{A_n} = 0, \quad M_r^{A_1 \dots A_{n-2} A_{n-1}}{}_{A_{n-1}} = -M_r^{A_1 \dots A_{n-2}} \quad (3.21)$$

Equation (3.21)₁ shows that the $M_r^{A_1 \dots A_n}$ have no components in direction of U^A ; they are projected.

By means of (3.3) we may decompose the moments (3.17) as

$$A_r^{A_1 A_2 \dots A_N} = \sum_{k=0}^N \binom{N}{k} \frac{1}{c^{N-k}} M_r^{(A_1 \dots A_k} U^{A_{k+1}} \dots U^{A_n)}, \quad (3.22)$$

where the brackets indicate symmetrization. On the other hand we may compute the PS moments from the unprojected moments with (3.5)₂,

$$M_r^{B_1 \dots B_k} = A_r^{A_1 \dots A_N} A_{A_1}^{B_1} \dots A_{A_k}^{B_k} \frac{1}{c^{N-k}} U_{A_{k+1}} \dots U_{A_N}. \quad (3.23)$$

3.4.3. PSTF moments. Because of (3.21)₂ only the trace-free parts of the $M_r^{A_1 \dots A_n}$ are independent, so it suffices to consider the trace-free parts only which we denote by

$$M_r^{\langle A_1 \dots A_n \rangle} = \int (p_{RL}^0)^{r+1} L^{\langle A_1} \dots L^{A_n \rangle} f dP, \quad r, n \in \mathbb{N}. \quad (3.24)$$

The $M_r^{\langle A_1 \dots A_n \rangle}$ have the properties

$$M_r^{\langle A_1 \dots A_n \rangle} U_{A_n} = 0, \quad M_r^{\langle A_1 \dots A_{n-1}}{}_{A_{n-1}} \rangle = 0. \quad (3.25)$$

In RLF they have spatial components only which we denote as

$$u_r^{\langle i_1 \dots i_n \rangle} = (M_r^{\langle i_1 \dots i_n \rangle})_{RL} = \left(\frac{\hbar}{c}\right)^{r+3} \int \omega^{r+2} n^{\langle i_1 \dots i_n \rangle} f d\omega d\Omega. \quad (3.26)$$

The densities of radiative energy and momentum in RLF are given by $e = cu_1$ and $p^j = u_1^j$. u_0 is the photon number density.

The angular brackets indicate a completely projected symmetric trace-free tensor. A PSTF tensor of rank n is related to a PS tensor by

$$M_r^{\langle A_1 \dots A_n \rangle} = M_r^{A_1 \dots A_n} + \sum_{k=1}^{\|n/2\|} \frac{1}{\prod_{j=0}^{k-1} (2(n-j)-1)} \\ \times (\Delta^{A_1 A_2} \dots \Delta^{A_{2k-1} A_{2k}} M_r^{A_{2k+1} \dots A_n} + \dots + (P_{nk} \text{ terms})) \quad (3.27)$$

where

$$\left\| \frac{n}{2} \right\| = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ uneven.} \end{cases} \quad (3.28)$$

The sum in brackets extends over all

$$P_{nk} = \frac{n!}{(n-2k)! 2^k k!} \quad (3.29)$$

different permutations of the indices. The first few tensors $M_r^{\langle A_1 \dots A_n \rangle}$ read

$$M_r^{\langle A \rangle} = M_r^A \\ M_r^{\langle AB \rangle} = M_r^{AB} + \frac{1}{3} \Delta^{AB} M_r \\ M_r^{\langle ABC \rangle} = M_r^{ABC} + \frac{1}{5} (\Delta^{AB} M_r^C + \Delta^{AC} M_r^B + \Delta^{BC} M_r^A) \\ M_r^{\langle ABCD \rangle} = M_r^{ABCD} + \frac{1}{7} (\Delta^{AB} M_r^{CD} + \Delta^{AC} M_r^{BD} + \Delta^{AD} M_r^{BC} \\ + \Delta^{BC} M_r^{AD} + \Delta^{BD} M_r^{AC} + \Delta^{CD} M_r^{AB}) \\ + \frac{1}{35} (\Delta^{AB} \Delta^{CD} + \Delta^{AC} \Delta^{BD} + \Delta^{AD} \Delta^{BC}) M_r.$$

The projected moments (3.20) are related to the PSTF moments due to

$$M_r^{A_1 \dots A_n} = M_r^{\langle A_1 \dots A_n \rangle} + \sum_{k=1}^{\|n/2\|} \frac{(-1)^k}{\prod_{j=0}^{k-1} (2(n-k-j)+1)} \\ \times (\Delta^{A_1 A_2} \dots \Delta^{A_{2k-1} A_{2k}} M_r^{\langle A_{2k+1} \dots A_n \rangle} + \dots + (P_{nk} \text{ terms})). \quad (3.31)$$

With (3.22, 3.31) it is possible to represent the moments $A_r^{A_1 A_2 \dots A_N}$ through the PSTF moments $M_r^{\langle A_1 \dots A_n \rangle}$. On the other hand the PSTF moments may be computed from the $A_r^{A_1 A_2 \dots A_N}$ by use of the relations (3.23, 3.27).

From the formulae above one derives two relations which will be needed later, viz.

$$M_r^{\langle B_1 \dots B_n \rangle C} = M_r^{\langle B_1 \dots B_n C \rangle} - \frac{n}{2n+1} M_r^{\langle\langle B_1 \dots B_{n-1} \rangle \Delta^{B_n} \rangle C} \quad (3.32)$$

$$\begin{aligned} M_r^{\langle B_1 \dots B_n \rangle CD} &= M_r^{\langle B_1 \dots B_n CD \rangle} - \frac{n}{2n+3} (M_r^{\langle D \langle B_1 \dots B_{n-1} \rangle \Delta^{B_n} \rangle C} \\ &\quad + M_r^{\langle C \langle B_1 \dots B_{n-1} \rangle D \Delta^{B_n} \rangle D} - \frac{1}{2n+3} M_r^{\langle B_1 \dots B_n \rangle \Delta^{CD}} \\ &\quad + \frac{n(n-1)}{(2n+1)(2n-1)} \Delta^{C \langle B_1} M_r^{\langle B_2 \dots B_{n-1} \rangle \Delta^{B_n} \rangle D}. \end{aligned} \quad (3.33)$$

Note that the Eqs. (3.31, 3.27, 3.32, 3.33) are valid for all projected symmetric tensors.

3.5. Moment Equations

3.5.1. *Unprojected moments.* Multiplication of the radiative transfer Eq. (3.10) with

$$(p_{RL}^0)^{r-N} p^{A_1} p^{A_2} \dots p^{A_N} \frac{\sqrt{-g}}{p_0}$$

and subsequent integration over d^3p gives the equations for the unprojected moments $A_r^{A_1 A_2 \dots A_N}$. After some partial integrations and with use of the identities

$$p^A{}_{;B} = \Gamma_{BC}^a p^C \frac{\partial p^A}{\partial p^a}, \quad (\ln \sqrt{-g})_{;A} = \Gamma_{AB}^B$$

we obtain

$$A_r^{A_1 A_2 \dots A_N B}{}_{;B} + (N-r) A_r^{A_1 A_2 \dots A_N B} \frac{1}{c} U_{B;D} = \mathcal{P}_r^{A_1 A_2 \dots A_N}. \quad (3.34)$$

The production tensor $\mathcal{P}_r^{A_1 A_2 \dots A_N}$ is defined by

$$\mathcal{P}_r^{A_1 A_2 \dots A_N} = \int (p_{RL}^0)^{r-N} p^{A_1} p^{A_2} \dots p^{A_N} \frac{p_{RL}^0}{c} \mathcal{S} dP. \quad (3.35)$$

3.5.2. *PSTF moments* The compact Eqs. (3.34) are not well suited for calculation. Here one needs the equations for the PSTF moments. Multiplication of (3.34) for $N = n$ with $\Delta_{A_1} \langle B_1 \dots B_n \rangle \dots \Delta_{A_n} \langle B_1 \dots B_n \rangle$ yields after some rearrangement

$$\begin{aligned}
& DM_r \langle B_1 \dots B_n \rangle + \nabla_C M_r \langle B_1 \dots B_n \rangle^C \\
& + \frac{1}{c} DU_D \left\{ (n-r-1) M_r \langle B_1 \dots B_n \rangle^D + ng^{D \langle B_1} M_r \langle B_2 \dots B_n \rangle \rangle + n \frac{1}{c} U \langle B_1 M_r \langle B_2 \dots B_n \rangle \rangle^D \right\} \\
& + \frac{1}{c} \nabla_C U_D \left\{ (n-r) M_r \langle B_1 \dots B_n \rangle^{CD} + M_r \langle B_1 \dots B_n \rangle g^{CD} + ng^{D \langle B_1} M_r \langle B_2 \dots B_n \rangle \rangle^C \right. \\
& \left. + n \frac{1}{c} U \langle B_1 M_r \langle B_2 \dots B_n \rangle \rangle^{CD} \right\} = P_r \langle B_1 \dots B_n \rangle. \tag{3.36}
\end{aligned}$$

The production tensor is defined as

$$\begin{aligned}
P_r \langle B_1 \dots B_n \rangle &= \Delta_{A_1} \langle B_1 \dots \Delta_{A_n} \langle B_n \rangle \mathcal{P}_r^{A_1 \dots A_n} \\
&= \frac{1}{c} \int (p_{RL}^0)^{r+1} L \langle B_1 \dots L \langle B_n \rangle \mathcal{P} dP. \tag{3.37}
\end{aligned}$$

The abbreviations D and ∇_C stand for the covariant generalizations of the partial derivatives in RLF with respect to time and space, respectively,

$$D\Psi = \frac{1}{c} U^C \Psi_{;C}, \quad \nabla_C \Psi = \Delta_C^D \Psi_{;D}$$

so that

$$\Psi_{;C} = \frac{1}{c} U_C D\Psi + \nabla_C \Psi. \tag{3.38}$$

Equation (3.36) is not formulated for the completely trace-free moments. Here one has to use the identities (3.32, 3.33). We do not make this insertion at this stage. For the case $r = 1$ the Eqs. (3.36) were written for the first time by Thorne [22]. The Eqs. (3.36) for $n = 0, 1, \dots, N$ are equivalent to (3.34).

4. CONSTITUTIVE EQUATIONS

4.1. Variables and Closure Problem

We assume that the knowledge of the moments

$$M_r \langle A_1 \dots A_n \rangle, \quad r = 0, 1, \dots, R; \quad n = 0, 1, \dots, N$$

or

$$A_r^{A_1 \cdots A_N}, \quad r=0, 1, \dots, R \quad (4.1)$$

describes a given radiative process with sufficient accuracy. For the moment R, N are arbitrary numbers. Of course we have to answer the question which values of R, N must be chosen for a radiative process. At the end of the paper we shall give some criteria for the determination of R . A subsequent paper [21] will discuss the number N .

The moments (4.1) must be determined from the Eqs. (3.36) with $r=0, \dots, R$; $n=0, \dots, N$. These equations do not form a closed set for the variables $M_r^{\langle A_1 \cdots A_n \rangle}$ because they contain the additional moments $M_r^{\langle A_1 \cdots A_n B \rangle}$, $M_r^{\langle A_1 \cdots A_n BC \rangle}$ and the productions $P_r^{\langle A_1 \cdots A_n \rangle}$. The closure requires constitutive equations which relate these quantities to the variables (4.1). We choose constitutive equations which depend on the variables (4.1)

$$\begin{aligned} P_r^{\langle A_1 \cdots A_n \rangle} &= P_r^{\langle A_1 \cdots A_n \rangle} (M_s^{\langle B_1 \cdots B_m \rangle}, s=0, \dots, R; m=0, \dots, N), \\ M_r^{\langle A_1 \cdots A_n B \rangle} &= M_r^{\langle A_1 \cdots A_n B \rangle} (M_s^{\langle B_1 \cdots B_m \rangle}, s=0, \dots, R; m=0, \dots, N), \\ M_r^{\langle A_1 \cdots A_n BC \rangle} &= M_r^{\langle A_1 \cdots A_n BC \rangle} (M_s^{\langle B_1 \cdots B_m \rangle}, s=0, \dots, R; m=0, \dots, N). \end{aligned} \quad (4.2)$$

Note that gradients or time derivatives are absent from the list of variables. This choice of variables is motivated by the theory of Extended Thermodynamics of Müller and Ruggeri [15]. Indeed, we will use the entropy maximum principle [7] for closure which is equivalent to Extended Thermodynamics. Thus all features of Extended Thermodynamics will be contained in our theory. In particular the resulting field equations will be of symmetric hyperbolic type. This guarantees well-posedness of Cauchy problems and finite speeds of disturbances.

4.2. Entropy Maximum Principle

The definitions (3.20, 3.37) show that we will find the required constitutive Eqs. (4.2) if the photon phase density f depends on space-time only through the variables,

$$f = f(M_s^{\langle B_1 \cdots B_m \rangle}(x^A), p^i). \quad (4.3)$$

The entropy maximum principle states: The phase density (4.3) follows by maximization of the entropy density in RLF with respect to f under the constraint of prescribed values of the variables $M_s^{\langle B_1 \cdots B_m \rangle}$. The entropy density in RLF is given by

$$h_{LR} = \frac{1}{c^2} S^A U_A = -k_B \int p_{RL}^0 \left[f \ln \frac{f}{y} - (y + f) \ln \left(1 + \frac{f}{y} \right) \right] dP. \quad (4.4)$$

We take care of the constraints by Lagrange multipliers $A^r_{\langle A_1 \dots A_n \rangle}$ and maximize

$$-k_B \int p_{RL}^0 \left[f \ln \frac{f}{y} - (y+f) \ln \left(1 + \frac{f}{y} \right) \right] dP - \sum_{r,n} A^r_{\langle A_1 \dots A_n \rangle} \left[\int (p_{RL}^0)^{r+1} L^{\langle A_1 \dots A_n \rangle} f dP - M_r^{\langle A_1 \dots A_n \rangle} \right] \quad (4.5)$$

without constraints. The result reads

$$f = \frac{y}{\exp \Sigma - 1}, \quad \text{with} \quad \Sigma = \frac{1}{k_B} \sum_{r,n} A^r_{\langle A_1 \dots A_n \rangle} (p_{RL}^0)^r L^{\langle A_1 \dots A_n \rangle} \quad (4.6)$$

The Lagrange multipliers are PSTF tensors too and must be calculated from the definition (3.20) of the moments.

4.3. Expansion of the Phase Density

Unfortunately it is impossible to perform the required integrals over the phase density (4.6). For this reason we will expand it around equilibrium. Comparison with the equilibrium phase density (3.12) shows that we have for the Lagrange multipliers in equilibrium

$$\sum_{r,n} A^r_{\langle A_1 \dots A_n \rangle | E} = \begin{cases} \frac{c}{T}, & r=1, \quad n=0 \\ 0, & \text{else.} \end{cases} \quad (4.7)$$

We write

$$\frac{1}{k_B} A^r_{\langle A_1 \dots A_n \rangle} = \frac{c}{k_B T} \delta^{r1} \delta_{n0} + \lambda^r_{\langle A_1 \dots A_n \rangle} \quad (4.8)$$

and assume that the non-equilibrium parts $\lambda^r_{\langle A_1 \dots A_n \rangle}$ of the Lagrange multipliers are small such that

$$f \approx f_{|E} + \frac{df_{|E}}{d\Sigma_{|E}} \sum_{r,n} \lambda^r_{\langle A_1 \dots A_n \rangle} (p_{RL}^0)^r L^{\langle A_1 \dots A_n \rangle}. \quad (4.9)$$

The phase density is a series in the powers $(p_{RL}^0)^r$ and the spherical harmonics $L^{\langle A_1 \dots A_n \rangle}$. The powers reflect the spectral deviation from the equilibrium phase density while the spherical harmonics account for anisotropy. If the deviation from equilibrium is too big, this expansion is not appropriate. For instance it will be inappropriate for a sunbeam travelling into the earth' atmosphere. This problem will be discussed in [21].

4.4. Moments and Lagrange-Multipliers

The calculation of the Lagrange multipliers and the constitutive equations is best performed in RLF. The RLF Lagrange multipliers follow from (3.26)

$$\lambda_{\langle i_1 i_2 \dots i_n \rangle}^t = -\frac{\prod_{j=0}^n (2j+1)}{n!} \sum_r \mathcal{C}_{tr}^{-1} \frac{u_r^{\langle i_1 i_2 \dots i_n \rangle} - u_r^{\langle i_1 i_2 \dots i_n \rangle|E}}{4\pi y (k_B T/c)^{r+t+3}}, \quad (4.10)$$

where we have introduced the equilibrium values of the moments

$$u_{r|E}(T) = M_{r|E}(T) = 4\pi y \left(\frac{k_B T}{c} \right)^{r+3} \Gamma(r+3) \zeta(r+3), \quad (4.11)$$

$$u_r^{\langle i_1 \dots i_n \rangle|E}(T) = 0,$$

and a matrix defined as

$$\mathcal{C}_{rs} = C(r, s), \quad r, s = 0, \dots, R$$

with

$$C(r, s) = \Gamma(s+r+3) \zeta(s+r+2). \quad (4.12)$$

$\Gamma(x)$ denotes the Gamma function and $\zeta(x)$ is Riemann's Zeta function.

4.5. Constitutive Equations for Moments

With (4.9, 4.10) we have found the phase density (4.3) and are able to calculate the constitutive equations. In RLF we obtain in general

$$u_r^{\langle i_1 \dots i_{N+m} \rangle} = 0, \quad m > 0. \quad (4.13)$$

For the moments (4.2)_{2,3} follows

$$M_r^{\langle A_1 \dots A_N B \rangle} = 0, \quad M_r^{\langle A_1 \dots A_N B D \rangle} = 0. \quad (4.14)$$

The PSTF moments with tensorial rank $> N$ vanish. Since the number N reflects the anisotropy of the photon field, N must be chosen big enough, so that Eqs. (4.14) are approximately valid, see [21].

4.6. Constitutive Equations for Productions

For the productions (3.37) we obtain

$$P_r = -\sum_s \frac{1}{c} \Theta_{rs} (M_s - M_{s|E}), \quad (4.15)$$

$$P_r^{\langle A_1 \dots A_n \rangle} = -\sum_s \frac{1}{c} \hat{\Theta}_{rs} M_s^{\langle A_1 \dots A_n \rangle}, \quad n \neq 2 \quad (4.16)$$

$$P_r^{\langle A_1 A_2 \rangle} = -\sum_s \frac{1}{c} \left(\frac{9}{10} \hat{\Theta}_{rs} + \frac{1}{10} \Theta_{rs} \right) M_s^{\langle A_1 A_2 \rangle}. \quad (4.17)$$

In the case of isotropic scattering (3.15) the production (4.17) fits into the scheme (4.16),

$$P_r^{\langle A_1 A_2 \rangle} = - \sum_s \frac{1}{c} \hat{\Theta}_{rs} M_s^{\langle A_1 A_2 \rangle}. \quad (4.18)$$

Θ_{rs} denotes the matrix of mean absorption coefficients defined as

$$\Theta_{rs}(\varrho, T) = \left(\frac{k_B T}{c} \right)^{r-s} \sum_t \mathcal{G}_{ts}^{-1} \int \kappa(\Sigma) \frac{e^\Sigma}{(e^\Sigma - 1)^2} \Sigma^{t+r+2} d\Sigma \quad (4.19)$$

and the matrix of mean absorption and scattering coefficients $\hat{\Theta}_{rs}$ is defined as

$$\hat{\Theta}_{rs}(\varrho, T) = \Theta_{rs}(\varrho, T) + \left(\frac{k_B T}{c} \right)^{r-s} \sum_t \mathcal{G}_{ts}^{-1} \int \zeta(\Sigma) \frac{e^\Sigma}{(e^\Sigma - 1)^2} \Sigma^{t+r+2} d\Sigma. \quad (4.20)$$

Only if scattering is negligible, is it possible to find an easy form for the productions (3.35), viz.

$$\mathcal{P}_r^{A_1 A_2 \dots A_N} = - \sum_s \frac{1}{c} \Theta_{rs} (A_s^{A_1 A_2 \dots A_N} - A_s^{A_1 A_2 \dots A_N}|_E). \quad (4.21)$$

4.7. Bremsstrahlung Absorption and Thomson Scattering

In the case of bremsstrahlung absorption the matrix of mean absorption coefficients comes out as

$$\Theta_{rs}^{ff} = D_{ff} \left(\frac{k_B T}{c} \right)^{r-s} \sum_t \Gamma(t+r) \zeta(t+r) \mathcal{G}_{ts}^{-1}. \quad (4.22)$$

Because of $\zeta(0) = \infty$ the matrix Θ_{rs}^{ff} becomes indeterminate for $r = t = 0$. This problem can be avoided, if the variables with index $r = 0$ are deleted from the list of variables.

We show why this must be done: The net number rate of photons with 3-momentum p^i created by bremsstrahlung processes is given by

$$dN = c \left(\frac{p_{RL}^0}{c} \mathcal{F} \right) dP. \quad (4.23)$$

For the RLF observer this may be written with (3.11, 3.13, 4.9) as

$$dN = -D_{ff} \left(\frac{k_B T}{c} \right)^3 \left(\frac{\hbar}{c} \right)^r \sum_{r,n} \lambda_{\langle i_1 \dots i_n \rangle}^r f_{|E} \omega^{r-1} n^{\langle i_1 \dots i_n \rangle} d\omega d\Omega. \quad (4.24)$$

This number must be finite for all frequencies, especially for $\omega \rightarrow 0$. It follows that

$$\lambda_{\langle i_1 i_2 \dots i_n \rangle}^0 = 0 \quad (4.25)$$

must hold for all n . This is tantamount to deleting the moments $M_0^{\langle A_1 \dots A_n \rangle}$ from the list of variables, the index r runs over the values $r = 1, \dots, R$ only.

In the case Thomson scattering (3.16) the matrix of mean absorption and scattering coefficients comes out as

$$\hat{\Theta}_{rs}(T) = \Theta_{rs} + \zeta_T \delta_{rs}. \quad (4.26)$$

5. THE EQUATIONS OF RADIATION THERMODYNAMICS

5.1. Coupling to Matter

With the results of the last section we are able to write down the closed set of field equations for radiative processes. They follow from (3.36) with (4.15–4.20). These field equations are coupled to the equations for matter (2.3, 2.4) by the energy momentum production vector P^A which is given by, see (3.19)

$$P^A = c\mathcal{P}_1^A = U^A P_1 + cP_1^A = -\frac{1}{c} U^A \sum_s \Theta_{1s} (M_s - M_{s|E}) - \sum_s \hat{\Theta}_{1s} M_s^A. \quad (5.1)$$

An additional coupling is due to the occurrence of the temperature and density of matter in the matrices Θ_{rs} , $\hat{\Theta}_{rs}$ and in the equilibrium moments.

The moment equations form a hierarchy of N equations for each index r . The R hierarchies are coupled through the matrices Θ_{rq} , $\hat{\Theta}_{rq}$ on the right hand sides. This coupling is unusual although not unknown in the kinetic theory of gases [19]. In the present theory of radiation thermodynamics it is necessary and essential for good results.

5.2. Matter at Rest

The field equations have their simplest form for matter that moves with homogeneous speed. The equations for the RLF observer read

$$\begin{aligned} \frac{\partial u_r}{\partial t} + c \frac{\partial u_r^k}{\partial x^k} &= -\sum_s \Theta_{rs} (u_s - u_{s|E}) \\ \frac{\partial u_r^i}{\partial t} + \frac{n}{2n+1} c \frac{\partial u_r}{\partial x^i} + c \frac{\partial u_r^{\langle ik \rangle}}{\partial x^k} &= -\sum_s \hat{\Theta}_{rs} u_s^i \\ \frac{\partial u_r^{\langle i_1 i_2 \rangle}}{\partial t} + \frac{n}{2n+1} c \frac{\partial u_r^{\langle i_1 \rangle}}{\partial x^{i_2}} + c \frac{\partial u_r^{\langle i_1 i_2 k \rangle}}{\partial x^k} &= -\sum_s \left(\frac{9}{10} \hat{\Theta}_{rs} + \frac{1}{10} \Theta_{rs} \right) u_s^{\langle i_1 i_2 \rangle} \\ \frac{\partial u_r^{\langle i_1 \dots i_n \rangle}}{\partial t} + \frac{n}{2n+1} c \frac{\partial u_r^{\langle\langle i_1 \dots i_{n-1} \rangle\rangle}}{\partial x^{i_n}} + c \frac{\partial u_r^{\langle i_1 \dots i_n k \rangle}}{\partial x^k} &= -\sum_s \hat{\Theta}_{rs} u_s^{\langle i_1 \dots i_n \rangle} \\ \frac{\partial u_r^{\langle i_1 \dots i_N \rangle}}{\partial t} + \frac{N}{2N+1} c \frac{\partial u_r^{\langle\langle i_1 \dots i_{N-1} \rangle\rangle}}{\partial x^{i_N}} &= -\sum_s \hat{\Theta}_{rs} u_s^{\langle i_1 \dots i_N \rangle}. \end{aligned} \quad (5.2)$$

5.3. The Limit of Non-Relativistic Velocity

In many cases there exists an observer for whom the velocity of matter is small compared to the velocity of light. We call this frame laboratory system (LS). We neglect the effects of gravitation and all terms of order $\mathcal{O}(v^2/c^2)$. Thus we have for the LS observer

$$\begin{aligned} U^A &= \{c, v_i\}, & U_A &= \{c, -v_i\} \\ D\Psi &= \frac{1}{c} \left(\frac{\partial \Psi}{\partial t} + v_k \frac{\partial \Psi}{\partial x^k} \right), & \nabla_C \Psi &= \left\{ \frac{v_k}{c} \frac{\partial \Psi}{\partial x^k}, \frac{\partial \Psi}{\partial x^i} + \frac{1}{c^2} v_i \frac{\partial \Psi}{\partial t} \right\}_C \\ \frac{1}{c} D U_A &= \left\{ 0, \frac{-1}{c^2} \frac{\partial v_i}{\partial t} \right\}_A, & \frac{1}{c} \nabla_C U_D &= \left\{ \begin{array}{cc} 0 & 0_j \\ 0_i & -\frac{1}{c} \frac{\partial v_i}{\partial x^j} \end{array} \right\}_{CD} \end{aligned} \quad (5.3)$$

Furthermore this observer measures the vector L^A as

$$\bar{L}^A = \left\{ \frac{v_l}{c} n_l, n_k \right\} \quad (5.4)$$

and finds the following relations between the moments $M_r^{\langle A_1 \dots A_n \rangle}$ and $u_r^{\langle i_1 \dots i_n \rangle}$:

$$\begin{aligned} M_r^{\langle i_1 \dots i_n \rangle} &= u_r^{\langle i_1 \dots i_n \rangle}, & M_r^{\langle i_1 \dots i_n \rangle k} &= u_r^{\langle i_1 \dots i_n \rangle k}, \\ M_r^{\langle i_1 \dots i_n \rangle kl} &= u_r^{\langle i_1 \dots i_n \rangle kl}, & M_r^{\langle i_1 \dots i_n \rangle 0} &= u_r^{\langle i_1 \dots i_n \rangle k} \frac{v_k}{c}, \\ M_r^{\langle i_1 \dots i_n \rangle 00} &= \mathcal{O} \left(\frac{v^2}{c^2} \right). \end{aligned} \quad (5.5)$$

Consideration of the space components of (3.36) only, i.e., setting $B_k = i_k$ yields

$$\begin{aligned} & \frac{\partial u_r^{\langle i_1 \dots i_n \rangle}}{\partial t} + \frac{\partial}{\partial x^k} (u_r^{\langle i_1 \dots i_n \rangle} v^k + c u_r^{\langle i_1 \dots i_n \rangle k}) + (r-n) u_r^{\langle i_1 \dots i_n \rangle kl} \frac{\partial v_k}{\partial x^l} \\ & + n \frac{\partial v^{\langle i_1}}{\partial x^k} u_r^{\langle i_1 \dots i_n \rangle k} + \frac{v_k}{c} \frac{\partial u_r^{\langle i_1 \dots i_n \rangle k}}{\partial t} \\ & + (r+1-n) u_r^{\langle i_1 \dots i_n \rangle k} \frac{1}{c} \frac{\partial v_k}{\partial t} + n \frac{1}{c} \frac{\partial v^{\langle i_1}}{\partial t} u_r^{\langle i_1 \dots i_n \rangle} \\ & = c P_r^{\langle i_1 \dots i_n \rangle} + \mathcal{O} \left(\frac{v^2}{c^2} \right). \end{aligned} \quad (5.6)$$

These equations may be recast to give equations of balance type,

$$\frac{\partial m_r^{\langle i_1 \dots i_n \rangle}}{\partial t} + c \frac{\partial m_r^{\langle i_1 \dots i_n \rangle k}}{\partial x^k} = c p_r^{\langle i_1 \dots i_n \rangle} + \mathcal{O} \left(\frac{v^2}{c^2} \right) \quad (5.7)$$

with the PSTF moments and productions in LS given by

$$m_r^{\langle i_1 \cdots i_n \rangle} = u_r^{\langle i_1 \cdots i_n \rangle} + (r+1-n) u_r^{\langle i_1 \cdots i_n \rangle k} \frac{v_k}{c} + n \frac{v^{\langle i_1}}{c} u_r^{i_2 \cdots i_n \rangle} \quad (5.8)$$

$$p_r^{\langle i_1 \cdots i_n \rangle} = P_r^{\langle i_1 \cdots i_n \rangle} + (r-n) P_r^{\langle i_1 \cdots i_n \rangle k} \frac{v_k}{c} + n \frac{v^{\langle i_1}}{c} P_r^{i_2 \cdots i_n \rangle}. \quad (5.9)$$

6. LOCAL RADIATIVE EQUILIBRIUM

The above set of Eqs. (5.6) describes radiation interacting with matter in local radiative non-equilibrium. We investigate it in the limit of local radiative equilibrium (LRE) and compare it with the classic theory of LRE due to Rosseland [4].

Radiation is in LRE with matter if the photon distribution is a local Planck distribution. This is the case, if the mean free path of radiation (c/κ) is of the same order as the mean free path of matter.

We obtain the LRE equations by introducing the equilibrium values of the moments (4.11) into the left hand sides of (5.6), a procedure is known as Maxwellian iteration [16]. We ignore scattering and obtain for $n=0, 1, 2$ the equations

$$\begin{aligned} \frac{\partial u_{r|E}(T)}{\partial t} + v^k \frac{\partial u_{r|E}(T)}{\partial x^k} + \frac{r+3}{3} u_{r|E}(T) \frac{\partial v^k}{\partial x^k} &= -\sum_s \Theta_{rs}(T) (u_s - u_{s|E}(T)) \\ \frac{c}{3} \frac{\partial u_{r|E}(T)}{\partial x_i} + \frac{v^i}{c} \frac{1}{3} \frac{\partial u_{r|E}(T)}{\partial t} + \frac{r+3}{3} u_{r|E}(T) \frac{1}{c} \frac{\partial v^i}{\partial t} &= -\sum_s \Theta_{rs}(T) u_s^i \\ \frac{2}{5} \frac{r+3}{3} u_{r|E}(T) \frac{\partial v^{\langle i}}{\partial x_{j \rangle}} &= -\sum_s \Theta_{rs}(T) u_s^{\langle ij \rangle}. \end{aligned} \quad (6.1)$$

We use (4.11) to solve these equations for the radiative energy density $e = cu_1$, the radiative momentum density $p^i = u_1^i$ and the radiative pessusre tensor $N^{\langle ij \rangle} = cu_1^{\langle ij \rangle}$. This leads to ($e_{|E} = cu_{1|E} = aT^4$)

$$\begin{aligned} e &= e_{|E} - \frac{1}{\tilde{\kappa}} \left(\frac{\partial e_{|E}}{\partial t} + v^k \frac{\partial e_{|E}}{\partial x^k} + \frac{4}{3} e_{|E} \frac{\partial v_k}{\partial x^k} \right) \\ p_1^i &= -\frac{1}{3} \frac{1}{\tilde{\kappa}} \left(\frac{\partial e_{|E}}{\partial x_i} + \frac{v^i}{c^2} \frac{\partial e_{|E}}{\partial t} + 4e_{|E} \frac{\partial v^i}{\partial t} \right) \\ N^{\langle ij \rangle} &= -\frac{8}{15} \frac{1}{\tilde{\kappa}} e_{|E}(T) \frac{\partial v^{\langle i}}{\partial x_{j \rangle}}, \end{aligned} \quad (6.2)$$

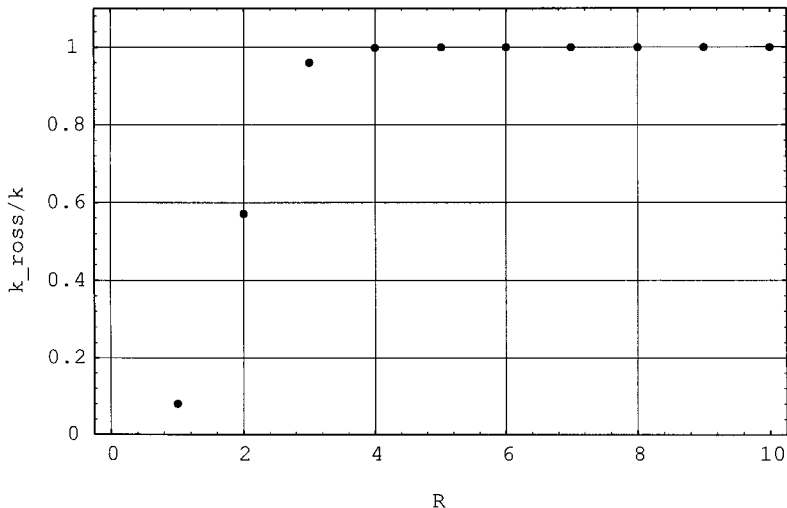


FIG. 6.1. $\tilde{\kappa}/\bar{\kappa}$ over the number of moments R for bremsstrahlung.

where we have introduced a mean absorption coefficient

$$\frac{1}{\tilde{\kappa}} = \sum_r \Theta_{1r}^{-1} \frac{\Gamma(r+4) \zeta(r+3)}{\Gamma(5) \zeta(4)} \left(\frac{k_B T}{c} \right)^{r-1}. \quad (6.3)$$

The same Eqs. (6.2) follow from the standard Rosseland procedure if $\tilde{\kappa}$ is replaced by the Rossel and mean

$$\frac{1}{\bar{\kappa}} = \frac{\int \frac{1}{\kappa(\Sigma_{|E})} \Sigma_{|E}^4 \frac{\partial f_{|E}}{\partial \Sigma_{|E}} d\Sigma_{|E}}{\int \Sigma_{|E}^4 \frac{\partial f_{|E}}{\partial \Sigma_{|E}} d\Sigma_{|E}}, \quad (6.4)$$

see [17], p. 121. The Rosseland mean is widely accepted to be the correct mean absorption coefficient in the case of LRE [14]. Thus the mean absorption coefficient $\tilde{\kappa}$ must be compared to the Rosseland mean $\bar{\kappa}$. We restrict ourselves to pure bremsstrahlung absorption and obtain

$$\frac{1}{\tilde{\kappa}} = \frac{1}{D_{ff}} \frac{\sum_{q,r} \Gamma(q+4) \zeta(q+3) A_{qr}^{-1} \Gamma(r+4) \zeta(r+3)}{\Gamma(5) \zeta(4)}, \quad (6.5)$$

$$\frac{1}{\bar{\kappa}} = \frac{1}{D_{ff}} \frac{\int \frac{\Sigma_{|E}^7 d\Sigma_{|E}}{(1 - e^{-\Sigma_{|E}})^2 (e^{\Sigma_{|E}} - 1)}}{\Gamma(5) \zeta(4)}$$

with a matrix

$$A_{rs} = \Gamma(r+s) \zeta(r+s). \quad (6.6)$$

Figure 6.1 shows the ratio $\check{\kappa}/\tilde{\kappa}$ for different values of the number of moments R . For $R \geq 4$ both coefficients agree, $\tilde{\kappa}$ converges fast to its limit $\check{\kappa}$. We conclude that our new theory contains the Rosseland mean in the limit of LRE for $R \geq 4$. Note, however, that $R=1$ gives a bad result.

Here we have found a first criterion for the number R : R must be chosen so that the mean absorption coefficient $(6.2)_2$ is almost equal to the Rosseland mean (6.4).

7. HOMOGENEOUS COMPRESSION OF RADIATION

7.1. A Simple Experiment

Considering a simple example we prove that the new theory gives good results in non-LRE too. We consider an experiment which may be evaluated by the radiative transfer equation as well as by the moment equations: the homogeneous compression of radiation and a gas. The experiment may be artificial from a practical point of view but it is instructive.

The experiment is performed in the following way (see Fig. 7.1): We consider a cylinder which is closed by a movable piston. Piston and cylinder are coated with ideal mirrors which guarantee that no radiation is absorbed or emitted from the walls. The cylinder contains a large number of scattering particles and a small number of absorbing particles which we describe as an monatomic ideal gas.

Starting from an equilibrium situation with temperature T_0 the piston is moved into the cylinder, the volume decreases from V_0 to V_1 . The motion should be fast enough so that the number of absorption processes during the motion is negligibly small. Because of the large number of scattering particles ($\zeta \gg \kappa$) the photon distribution will be homogeneous and isotropic at all times.

Furthermore it is assumed that the collision frequency of the gas is large compared to the absorption coefficient. Thus the compression of the gas follows the adiabatic law

$$T_1 = T_0 \left(\frac{V_0}{V_1} \right)^{2/3}. \quad (7.1)$$

Now the system is brought into contact with a heat reservoir of temperature T_1 and it is left on its own. Since the gas is always in equilibrium with itself the gas temperature does not change, it is equal to T_1 and, due to absorption processes, a relaxation of the non-equilibrium between radiation and gas will start.

For this process to equilibrium the radiative transfer Eq. (3.10) reduces to

$$\frac{df}{dt} = -\kappa(f - f_{|E}(T_1)), \quad (7.2)$$

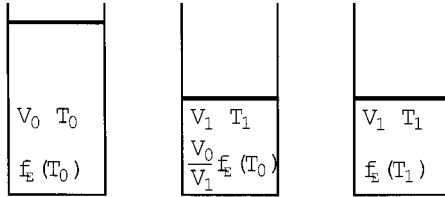


FIG. 7.1. Experimental setting for homogeneous compression of radiation and ideal gas.

while the moment equations (5.2) read

$$\frac{du^r}{dt} = -\sum_q \Theta_{rq}(T_1)(u^r - u_{|E}^r(T_1)), \quad r = 0, 1, \dots, R. \quad (7.3)$$

We ask for the energy density u^1 as a function of time. We will calculate u^1 by three different methods: (i) exact solution of the radiative transfer Eq. (7.2), (ii) two older simple moment methods which rely on the Planck mean and on the Rosseland mean, respectively, and (iii) the new moment method developed in this paper. The results of the moment theories will be compared with the exact solution (i)

7.2. Solution of the Radiative Transfer Equation

The photon distribution directly after the compression is

$$f_0 = \frac{V_0}{V_1} f_{|E}(T_0). \quad (7.4)$$

With this initial data we obtain from (7.2)

$$f(t) = f_{|E}(T_1) + \left(\frac{V_0}{V_1} f_{|E}(T_0) - f_{|E}(T_1) \right) e^{-\kappa t}. \quad (7.5)$$

A suitable measure for the approach of the moments towards equilibrium are the quantities

$$v^r = \frac{u^r - u_{|E}^r(T_1)}{(V_0/V_1) u_{|E}^r(T_0) - u_{|E}^r(T_1)}. \quad (7.6)$$

Directly after the compression we have $v^r(t=0) = 1$ while in equilibrium $v_{|E}^r = 0$ holds. With (7.5) and $u^r = (\hbar/c)^{r+3} \int \omega^{r+2} f d\omega d\Omega$ we obtain with the substitutions $x = \hbar\omega/k_B T_0$, $y = \hbar\omega/k_B T_1$

$$v^r(\tilde{t}) = \frac{\frac{V_0}{V_1} \left(\frac{T_0}{T_1} \right)^{r+3} \int \frac{x^{r+2}}{e^x - 1} e^{-\tilde{\kappa}((T_0/T_1)x)} \tilde{t} dx - \int \frac{y^{r+2}}{e^y - 1} e^{-\tilde{\kappa}(y)\tilde{t}} dy}{\Gamma(r+3) \zeta(r+3) \left[\frac{V_0}{V_1} \left(\frac{T_0}{T_1} \right)^{r+3} - 1 \right]}. \quad (7.7)$$

Here, we have introduced a new time scale \tilde{t} , so that

$$\tilde{\kappa}(y) \tilde{t} = \tilde{\kappa}(y) \check{\kappa}(T_1) t = \kappa t; \quad (7.8)$$

in the case of bremsstrahlung absorption we have

$$\tilde{\kappa}_{ff}(y) = \frac{1 - e^{-y}}{y^3}, \quad \check{\kappa}_{ff}(T_1) = D_{ff}. \quad (7.9)$$

(7.7) must be evaluated numerically. Figure 7.2 shows v^1 as function of \tilde{t} in the case of bremsstrahlung absorption. For this and the following calculations the ratio of volumes was chosen as $V_0/V_1 = 4$.

7.3. Planck Mean

Due to Thorne's theory [22] the energy density under the stated assumptions should obey the equation

$$\frac{du^1}{dt} = -\hat{\kappa}(u^1 - u^1_{|E}(T_1)), \quad (7.10)$$

where $\hat{\kappa}$ is the Planck mean of the absorption coefficient [22],

$$\hat{\kappa} = \frac{\int \kappa \Sigma^3_{|E} f_{|E} d\Sigma_{|E}}{\int \Sigma^3_{|E} f_{|E} d\Sigma_{|E}}. \quad (7.11)$$

In the case of bremsstrahlung we obtain

$$\hat{\kappa}_{ff} = \frac{D_{ff}}{\Gamma(4) \zeta(4)} = \frac{\check{\kappa}_{ff}(T_1)}{\Gamma(4) \zeta(4)} \quad (7.12)$$

and

$$v^1(\tilde{t}) = \exp\left(-\frac{\tilde{t}}{\Gamma(4) \zeta(4)}\right). \quad (7.13)$$

Figure 7.3 compares the predicted decay of v^1 obtained from the exact solution (7.7) and from (7.13). The Planck mean predicts a much faster decay. The difference is large and we conclude that the method of the Planck mean should not be used in this context.

7.4. Rosseland Mean

Due to Schweizer [20] the energy density under the stated assumptions should obey the equation

$$\frac{du^1}{dt} = -\check{\kappa}(u^1 - u^1_{|E}(T_1)), \quad (7.14)$$

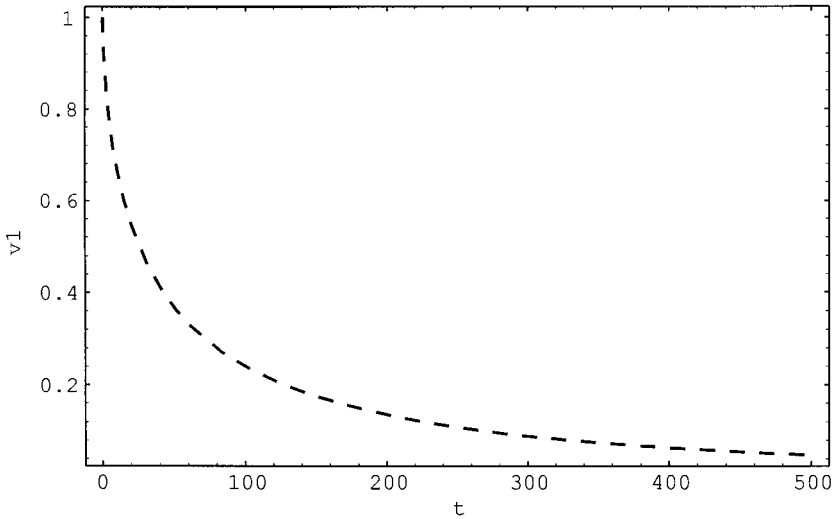


FIG. 7.2. Relaxation of v^1 in the case of bremsstrahlung absorption.

where $\check{\kappa}$ is the Rosseland mean of the absorption coefficient (6.4). In the case of bremsstrahlung we obtain

$$\check{\kappa}_{ff} = D_{ff} \frac{\Gamma(5) \zeta(4)}{5104.74} = \check{\kappa}_{ff}(T_1) \frac{\Gamma(5) \zeta(4)}{5104.74} \quad (7.15)$$

and

$$v^1(\tilde{t}) = \exp(-0.005089 \tilde{t}). \quad (7.16)$$

Figure 7.4 compares the predicted decay of v^1 obtained from the exact solution (7.7) and from (7.16).

The Rosseland mean predicts a much slower decay. The difference is large and we conclude that the method of the Rosseland mean should not be used in this context.

7.5. Solution of Moment Equations

We write the moment Equations (7.3) in terms of the quantities v^r (7.6) and \tilde{t} ,

$$\frac{dv^r}{d\tilde{t}} = -\sum_q \tilde{\Theta}_{rq} v^q \quad (7.17)$$

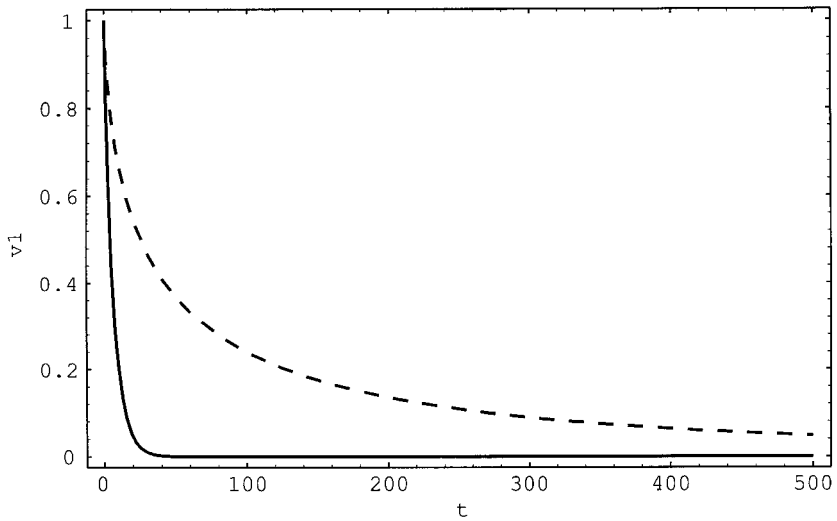


FIG. 7.3. $v^1(\bar{t})$ for bremsstrahlung from the radiative transfer equation (dotted line) and from the Rosseland mean (continuous line).

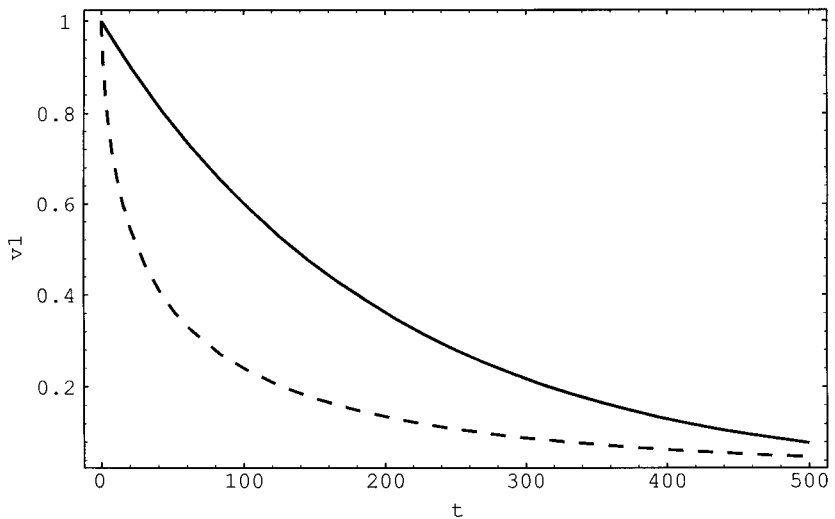


FIG. 7.4. $v^1(\bar{t})$ for bremsstrahlung from the radiative transfer equation (dotted line) and from the Rosseland mean (continuous line).

with the matrix (pure bremsstrahlung)

$$\tilde{\Theta}_{rq}^{ff} = \frac{\Gamma(q+3) \zeta(q+3)}{\Gamma(r+3) \zeta(r+3)} \frac{V_0 \left(\frac{T_0}{T_1}\right)^{q+3} - 1}{V_1 \left(\frac{T_0}{T_1}\right)^{r+3} - 1} \sum_s \Gamma(s+r) \zeta(s+r) \mathcal{C}_{sq}^{-1}. \quad (7.18)$$

The solution for v^r reads

$$v^r(\tilde{t}) = \sum_{p,s} M_{rp} M_{ps}^{-1} \exp -\alpha_p \tilde{t}, \quad (7.19)$$

where α_p are the eigenvalues of $\tilde{\Theta}_{rq}$ and M_{rp} is the matrix of its eigenvectors. Figure 7.5 shows the decay of v^1 calculated from the moment equations with the moments u^r , $r=1, \dots, R$ with $R=1$ and $R=6$ in comparison to the exact solution. Comparison with Figure 7.3 shows that the result is already better than the result obtained from the Planck mean.

For $R=6$ we find an excellent agreement between the solutions of the moment equations and the radiative transfer equation. We conclude that moments with subscript $r > 1$ must necessarily be considered.

This experiment gives a second criterion for the number R : It should be chosen so that the solution of the moment equations coincides with the solution of the radiative transfer equation. Note that in the process under consideration the spectral deviation from the equilibrium phase density is simple. It is likely that R must be increased in processes with more complicated initial data.

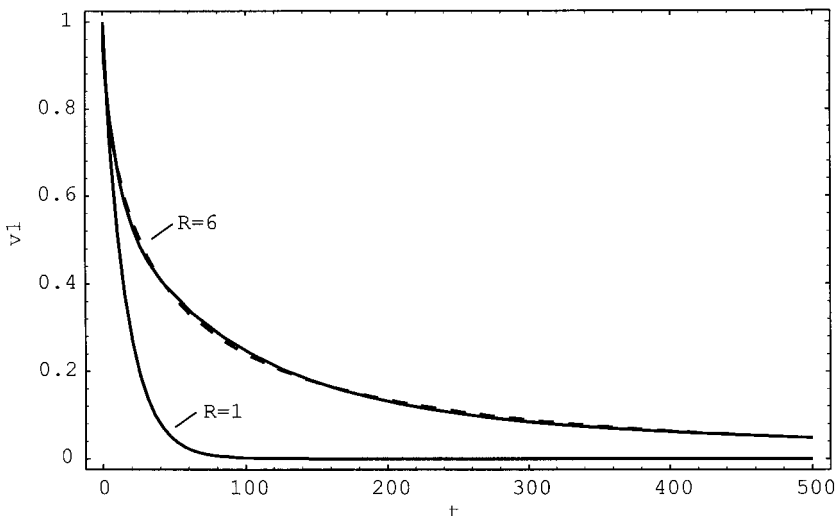


FIG 7.5. $v^1(\tilde{t})$ for bremsstrahlung from the radiative transfer equation (dotted line) and from the moment equations for $R=1, 6$ (continuous line).

8. CONCLUSION

The new theory of radiation thermodynamics with matrices of mean absorption and scattering coefficients presents itself as a powerful tool for the description of radiative processes. It reproduces the classical Rosseland mean in the LRE limit and—most important—describes non-LRE processes in coincidence with the radiative transfer equation. Since other theories fail already in the description of a simple homogeneous process, we suggest that the new theory should be used in non-LRE radiative transfer problems.

A detailed discussion for the inhomogeneous case of the new theory will be published in the near future [21]. There we will give an analytic criterion for the number of moments N . We will show that it depends on the ratio of the eigenvalues of the matrices and a typical frequency of the process under consideration. Moreover we will give an analytical solution for a one-dimensional radiative transfer process in an isothermal atmosphere at rest. Again, comparison with the radiative transfer equation will demonstrate the effectiveness of the new method.

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